Fraïssé Classes and the First-Order Limit Property

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Goal

For a finite relational language $\mathcal{L}$, fix a class $\mathcal{K}$ of finite $\mathcal{L}$-structures. For each n, let $\mathcal{K}_n$ be the set of members of $\mathcal{K}$ with universe $\{1, \ldots, n\}$, and sample M_n uniformly from $\mathcal{K}_n$.

Question. For every first-order sentence φ, does

$$\lim_{n \to \infty} \mathbb{P}(M_n \models \varphi)$$

exist?

Main result. In Fraïssé case, this is equivalent to checking convergence only on *finite conjunctions of extension axioms*.

Fraïssé classes and extension axioms

Fraïssé class

A class K of finite L-structures is a *Fraïssé class* if it satisfies:

- **Hereditary Property (HP).** If $A \in K$ and B is a substructure of A, then $B \in K$.
- **Joint Embedding Property (JEP).** For any $A, B \in K$, there exists $C \in K$ such that both A and B embed into C.
- **Amalgamation Property (AP).** For any embeddings $f : A \hookrightarrow B$ and $g : A \hookrightarrow C$ with $A, B, C \in K$, there exists $D \in K$ and embeddings $h : B \hookrightarrow D$, $k : C \hookrightarrow D$ such that $h \circ f = k \circ g$.

Joint Embedding Property (JEP)

For any $A, B \in K$, there exists $C \in K$ such that both embed into C.

$$\begin{array}{ccc} A & \searrow & \\ & & C \\ B & \nearrow & \end{array}$$

- Any two finite structures can be realized together inside a larger one.

Amalgamation Property (AP)

Given embeddings $f : A \hookrightarrow B$ and $g : A \hookrightarrow C$:

$$\begin{array}{ccccc} & & B & & \\ f & \nearrow & & \searrow & h \\ & A & \circlearrowleft & D & \\ g & \searrow & & \nearrow & k \\ & & C & & \end{array}$$

- D amalgamates B and C over A.
- This property underlies ultrahomogeneity of the Fraïssé limit.

Fraïssé limits

Theorem. Let K be a countable class of finite L-structures (up to isomorphism) satisfying **HP**, **JEP**, and **AP**.

There exists a countable L-structure M such that:

- $\mathrm{Age}(M) = K$;
- M is **ultrahomogeneous**: every isomorphism between finite substructures of M extends to an automorphism of M.

Moreover, M is unique up to isomorphism.

We call M the *Fraïssé limit* of K, denoted $\lim(K)$.

Existence

- Enumerate $K = \{A_0, A_1, A_2, \dots\}$ (up to isomorphism).
- Build an increasing chain
$$M_0 \leq M_1 \leq M_2 \leq \cdots$$
of finite structures in K.
- At stage s, use **JEP** and **AP** to ensure:
  - A_s embeds into some later M_t;
  - every partial isomorphism between finite substructures can be extended along the chain.
- Let $M = \bigcup_s M_s$. Then $\mathrm{Age}(M) = K$ and M is ultrahomogeneous.

Uniqueness

Suppose M and N are countable structures with

$$\mathrm{Age}(M) = \mathrm{Age}(N) = K \quad \text{and both are ultrahomogeneous.}$$

- Using a back-and-forth argument, build an isomorphism $M \cong N$:
  - **Forth:** extend a finite partial isomorphism by embedding the next element of M into N;
  - **Back:** symmetrically extend by embedding the next element of N into M.

Conclusion: the Fraïssé limit $\lim(K)$ is unique up to isomorphism.

Extension axioms

Given $A, B \in \mathcal{K}$ with a strong embedding $A \leq B$:

Extension axiom (from A to B) : Any copy of A extends to a copy of B.

This is an $\forall\exists$-sentence.

Extension axioms encode the ultrahomogeneity/extension property of $\lim(\mathcal{K})$.

Formalizing Extension axioms

Fix enumerations $A = \{a_1, \ldots, a_m\}$ and $B = \{a_1, \ldots, a_m, b_1, \ldots, b_t\}$ with $A \leq B$. The extension axiom $\xi_{A \to B}$ is:

$$\forall x_1 \cdots \forall x_m \Big(\mathrm{Iso}_A(\bar{x}) \ \to \ \exists y_1 \cdots \exists y_t \ \mathrm{Iso}_B(\bar{x}, \bar{y}) \Big).$$

Let

$$\mathrm{Ext}(\mathcal{K}) = \{\xi_{A \to B} : A, B \in \mathcal{K},\ A \leq B\}.$$

FO limit property vs. 0–1 laws

- **FO limit property:** for every $\varphi \in \mathrm{FO}(\mathcal{L})$,

$$\lim_{n\to\infty} \mathbb{P}(\mathcal{K}_n, \varphi) \text{ exists in } [0,1].$$

- **0–1 law:** the limit exists and is always in $\{0,1\}$.

Example. In Fraïssé classes with every finite graphs, the 0-1 law holds.
In Fraïssé class of forest of caterpillars, the FO limit property holds.
In class of caterpillars, there are first-order sentences φ such that

$$\lim_{n\to\infty} \mathbb{P}(\mathcal{K}_n, \varphi) = p \quad \text{for some } 0 < p < 1.$$

Main theorem

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Theorem (Criterion for FO limit property). Let $\mathcal{K}$ be a Fraïssé class in a finite relational language. Then $\mathcal{K}$ has the FO limit property *iff* for every finite set $\Sigma \subseteq \mathrm{Ext}(\mathcal{K})$ the limit

$$\lim_{n \to \infty} \mathbb{P}(\mathcal{K}_n, \bigwedge \Sigma)$$

exists.

Notes on Main Theorem

- Extension axioms are *local* combinatorial conditions.
- The theorem reduces convergence for *all* FO sentences to convergence on a generating family.
- Asymptotic first-order behavior is already encoded in $\mathrm{Ext}(\mathcal{K})$.

Methodology

Stone space

Let $U = \mathrm{Th}_\forall(\mathrm{Age}(\mathcal{K}))$ and let S be the Stone space of complete theories extending U.

- Each sentence θ corresponds to a clopen set $[\theta] \subseteq S$.
- Define measures μ_n on S by

$$\mu_n([\theta]) := \mathbb{P}(\mathcal{K}_n, \theta).$$

Goal. Prove $\mu_n([\varphi])$ converges for every φ.

Topological properties of the Stone space

Let S be the Stone space of complete theories extending $U = \mathrm{Th}_\forall(\mathrm{Age}(K))$.

Then S satisfies:

- **Compact**: every open cover has a finite subcover.
- **Hausdorff**: distinct points are separated by disjoint open sets.
- **Zero-dimensional**: S has a basis consisting of clopen sets.

These properties are standard consequences of Stone duality for Boolean algebras.

A clopen basis generated by extension axioms

Let

$$\mathcal{C} = \{[\xi] : \xi \in \mathrm{Ext}(K)\}.$$

- Each $[\xi]$ is clopen in S.
- $\mathcal{C}$ separates points of S.
- The Boolean algebra $\mathcal{B}$ generated by $\mathcal{C}$ forms a clopen basis of the topology on S.

These follows by that the theory $U \cup \mathrm{Ext}(\mathcal{K})$ is complete.
Thus, every open set in S can be written as a union of sets from $\mathcal{B}$.

Dual topology

- Probability measures μ_n are defined on clopen sets $[\varphi]$.
- To study convergence of measures, we test them against functions.
- The natural class of test functions is $C(S)$, the space of continuous functions.

Goal. Reduce the set of test functions by finding dense subalgebra of $C(S)$.

Stone–Weierstrass in the zero-dimensional setting

Theorem (Stone–Weierstrass). Let S be a compact Hausdorff, zero-dimensional space.

Let $\mathcal{A} \subseteq C(S)$ be a subalgebra such that:

- $\mathcal{A}$ contains the constant functions.
- $\mathcal{A}$ separates points of S.
- $\mathcal{A}$ is closed under multiplication.

Then $\mathcal{A}$ is dense in $C(S)$ with respect to the uniform norm.

Applying Stone–Weierstrass

Let $\mathcal{B}$ be the Boolean algebra of clopen sets generated by extension axioms, and define

$$\mathcal{A} := \mathrm{span}\{\mathbf{1}_B : B \in \mathcal{B}\} \subseteq C(S).$$

Then:

- $\mathcal{A}$ contains constant functions.
- $\mathcal{A}$ separates points of S.
- $\mathcal{A}$ is an algebra under pointwise operations.

Hence, by Stone–Weierstrass,

$$\overline{\mathcal{A}} = C(S).$$

Weak-* convergence from a dense test set

If

$$\int h\, d\mu_n \to \int h\, d\mu \qquad \text{for all } h \in \mathcal{A},$$

then $\mu_n \Rightarrow \mu$ weakly, hence

$$\int f\, d\mu_n \to \int f\, d\mu \qquad \text{for all } f \in C(S).$$

Main Theorem

Assumption: limits exist for $\mathbb{P}(\mathcal{K}_n, \bigwedge \Sigma)$ for all finite $\Sigma \subseteq \mathrm{Ext}(\mathcal{K})$.

- This gives convergence of $\mu_n(B)$ for all $B \in \mathcal{B}$.
- Hence convergence on $\mathcal{A}$, hence $\mu_n \Rightarrow \mu$.
- For any FO sentence φ, $[\varphi]$ is clopen, so

$$\mu_n([\varphi]) \to \mu([\varphi]).$$

Thus $\mathbb{P}(\mathcal{K}_n, \varphi)$ converges for all φ.

Conclusions and open problems

Conclusions and Applications

- For a Fraïssé class K, the first-order limit property holds *if and only if* all finite conjunctions of extension axioms have convergent probabilities.
- This reduces the asymptotic analysis of *all* first-order sentences to a finite, combinatorial generating family given by extension axioms.
- For the Fraïssé class of finite graphs, extension axioms are equivalent to extension properties of **RG**, which has limits. So, the class satisfies FO limit property.
- For the Fraïssé class of linear orders, extension axioms has limits, so it admits FO limit property.

Open problems and Further disscusions

1. (**0–1 law**) When do all limits lie in $\{0, 1\}$? Is there a Fraïssé class which satisfies FO limit property but not 0-1 law?
2. (**Generic 0–1 law**) When does the limiting behavior agree with $\mathrm{Th}(\lim(\mathcal{K}))$ (**0** if $\mathrm{Th}(\lim(\mathcal{K})) \models \neg\phi$ and **1** if $\mathrm{Th}(\lim(\mathcal{K})) \models \phi$)?

- In the class of caterpillars, the FO-limit property holds, but 0-1 law fails. (This class is not Fraïssé)
- In the class of forests of caterpillars, the FO-limit property and 0-1 law holds. This is shown by O. Heinig, T. M¨uller, M. Noy, and S. Taraz (2024)
- In the class of finite linear orders, the 0-1 law holds, but generic 0-1 law fails. (With the sentence encoding density property)

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Thank you!