

Fraïssé Classes and the First-Order Limit Property

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Goal

For a finite relational language \mathcal{L} , fix a class \mathcal{K} of finite \mathcal{L} -structures. For each n , let \mathcal{K}_n be the set of members of \mathcal{K} with universe $\{1, \dots, n\}$, and sample M_n uniformly from \mathcal{K}_n .

Question. For every first-order sentence φ , does

$$\lim_{n \rightarrow \infty} \mathbb{P}(M_n \models \varphi)$$

exist?

Main result. In Fraïssé case, this is equivalent to checking convergence only on *finite conjunctions of extension axioms*.

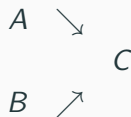
Fraïssé classes and extension axioms

A class K of finite L -structures is a *Fraïssé class* if it satisfies:

- **Hereditary Property (HP).** If $A \in K$ and B is a substructure of A , then $B \in K$.
- **Joint Embedding Property (JEP).** For any $A, B \in K$, there exists $C \in K$ such that both A and B embed into C .
- **Amalgamation Property (AP).** For any embeddings $f : A \hookrightarrow B$ and $g : A \hookrightarrow C$ with $A, B, C \in K$, there exists $D \in K$ and embeddings $h : B \hookrightarrow D$, $k : C \hookrightarrow D$ such that $h \circ f = k \circ g$.

Joint Embedding Property (JEP)

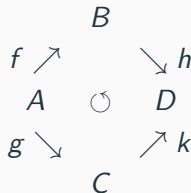
For any $A, B \in K$, there exists $C \in K$ such that both embed into C .



- Any two finite structures can be realized together inside a larger one.

Amalgamation Property (AP)

Given embeddings $f : A \hookrightarrow B$ and $g : A \hookrightarrow C$:



- D amalgamates B and C over A .
- This property underlies ultrahomogeneity of the Fraïssé limit.

Theorem. Let K be a countable class of finite L -structures (up to isomorphism) satisfying **HP**, **JEP**, and **AP**.

There exists a countable L -structure M such that:

- $\text{Age}(M) = K$;
- M is **ultrahomogeneous**: every isomorphism between finite substructures of M extends to an automorphism of M .

Moreover, M is unique up to isomorphism.

We call M the *Fraïssé limit* of K , denoted $\text{lim}(K)$.

- Enumerate $K = \{A_0, A_1, A_2, \dots\}$ (up to isomorphism).
- Build an increasing chain

$$M_0 \leq M_1 \leq M_2 \leq \dots$$

of finite structures in K .

- At stage s , use **JEP** and **AP** to ensure:
 - A_s embeds into some later M_t ;
 - every partial isomorphism between finite substructures can be extended along the chain.
- Let $M = \bigcup_s M_s$. Then $\text{Age}(M) = K$ and M is ultrahomogeneous.

Suppose M and N are countable structures with

$$\text{Age}(M) = \text{Age}(N) = K \quad \text{and both are ultrahomogeneous.}$$

- Using a back-and-forth argument, build an isomorphism $M \cong N$:
 - **Forth:** extend a finite partial isomorphism by embedding the next element of M into N ;
 - **Back:** symmetrically extend by embedding the next element of N into M .

Conclusion: the Fraïssé limit $\text{lim}(K)$ is unique up to isomorphism.

Given $A, B \in \mathcal{K}$ with a strong embedding $A \leq B$:

Extension axiom (from A to B) : Any copy of A extends to a copy of B .

This is an $\forall\exists$ -sentence.

Extension axioms encode the ultrahomogeneity/extension property of $\lim(\mathcal{K})$.

Formalizing Extension axioms

Fix enumerations $A = \{a_1, \dots, a_m\}$ and $B = \{a_1, \dots, a_m, b_1, \dots, b_t\}$ with $A \leq B$. The extension axiom $\xi_{A \rightarrow B}$ is:

$$\forall x_1 \cdots \forall x_m \left(\text{Iso}_A(\bar{x}) \rightarrow \exists y_1 \cdots \exists y_t \text{Iso}_B(\bar{x}, \bar{y}) \right).$$

Let

$$\text{Ext}(\mathcal{K}) = \{\xi_{A \rightarrow B} : A, B \in \mathcal{K}, A \leq B\}.$$

FO limit property vs. 0–1 laws

- **FO limit property:** for every $\varphi \in \text{FO}(\mathcal{L})$,

$$\lim_{n \rightarrow \infty} \mathbb{P}(\mathcal{K}_n, \varphi) \text{ exists in } [0, 1].$$

- **0–1 law:** the limit exists and is always in $\{0, 1\}$.

Example. In Fraïssé classes with every finite graphs, the 0-1 law holds.

In Fraïssé class of forest of caterpillars, the FO limit property holds.

In class of caterpillars, there are first-order sentences φ such that

$$\lim_{n \rightarrow \infty} \mathbb{P}(\mathcal{K}_n, \varphi) = p \quad \text{for some } 0 < p < 1.$$

Main theorem

Theorem (Criterion for FO limit property). Let \mathcal{K} be a Fraïssé class in a finite relational language. Then \mathcal{K} has the FO limit property *iff* for every finite set $\Sigma \subseteq \text{Ext}(\mathcal{K})$ the limit

$$\lim_{n \rightarrow \infty} \mathbb{P}(\mathcal{K}_n, \bigwedge \Sigma)$$

exists.

- Extension axioms are *local* combinatorial conditions.
- The theorem reduces convergence for *all* FO sentences to convergence on a generating family.
- Asymptotic first-order behavior is already encoded in $\text{Ext}(\mathcal{K})$.

Methodology

Let $U = \text{Th}_\forall(\text{Age}(\mathcal{K}))$ and let S be the Stone space of complete theories extending U .

- Each sentence θ corresponds to a clopen set $[\theta] \subseteq S$.
- Define measures μ_n on S by

$$\mu_n([\theta]) := \mathbb{P}(\mathcal{K}_n, \theta).$$

Goal. Prove $\mu_n([\varphi])$ converges for every φ .

Topological properties of the Stone space

Let S be the Stone space of complete theories extending $U = \text{Th}_\forall(\text{Age}(K))$.

Then S satisfies:

- **Compact**: every open cover has a finite subcover.
- **Hausdorff**: distinct points are separated by disjoint open sets.
- **Zero-dimensional**: S has a basis consisting of clopen sets.

These properties are standard consequences of Stone duality for Boolean algebras.

A clopen basis generated by extension axioms

Let

$$\mathcal{C} = \{[\xi] : \xi \in \text{Ext}(K)\}.$$

- Each $[\xi]$ is clopen in S .
- \mathcal{C} separates points of S .
- The Boolean algebra \mathcal{B} generated by \mathcal{C} forms a clopen basis of the topology on S .

These follows by that the theory $U \cup \text{Ext}(\mathcal{K})$ is complete.

Thus, every open set in S can be written as a union of sets from \mathcal{B} .

- Probability measures μ_n are defined on clopen sets $[\varphi]$.
- To study convergence of measures, we test them against functions.
- The natural class of test functions is $C(S)$, the space of continuous functions.

Goal. Reduce the set of test functions by finding dense subalgebra of $C(S)$.

Stone–Weierstrass in the zero-dimensional setting

Theorem (Stone–Weierstrass). Let S be a compact Hausdorff, zero-dimensional space.

Let $\mathcal{A} \subseteq C(S)$ be a subalgebra such that:

- \mathcal{A} contains the constant functions.
- \mathcal{A} separates points of S .
- \mathcal{A} is closed under multiplication.

Then \mathcal{A} is dense in $C(S)$ with respect to the uniform norm.

Applying Stone–Weierstrass

Let \mathcal{B} be the Boolean algebra of clopen sets generated by extension axioms, and define

$$\mathcal{A} := \text{span}\{\mathbf{1}_B : B \in \mathcal{B}\} \subseteq C(S).$$

Then:

- \mathcal{A} contains constant functions.
- \mathcal{A} separates points of S .
- \mathcal{A} is an algebra under pointwise operations.

Hence, by Stone–Weierstrass,

$$\overline{\mathcal{A}} = C(S).$$

Weak-* convergence from a dense test set

If

$$\int h \, d\mu_n \rightarrow \int h \, d\mu \quad \text{for all } h \in \mathcal{A},$$

then $\mu_n \Rightarrow \mu$ weakly, hence

$$\int f \, d\mu_n \rightarrow \int f \, d\mu \quad \text{for all } f \in C(S).$$

Main Theorem

Assumption: limits exist for $\mathbb{P}(\mathcal{K}_n, \bigwedge \Sigma)$ for all finite $\Sigma \subseteq \text{Ext}(\mathcal{K})$.

- This gives convergence of $\mu_n(B)$ for all $B \in \mathcal{B}$.
- Hence convergence on \mathcal{A} , hence $\mu_n \Rightarrow \mu$.
- For any FO sentence φ , $[\varphi]$ is clopen, so

$$\mu_n([\varphi]) \rightarrow \mu([\varphi]).$$

Thus $\mathbb{P}(\mathcal{K}_n, \varphi)$ converges for all φ .

Conclusions and open problems







Conclusions and Applications

- For a Fraïssé class K , the first-order limit property holds *if and only if* all finite conjunctions of extension axioms have convergent probabilities.
- This reduces the asymptotic analysis of *all* first-order sentences to a finite, combinatorial generating family given by extension axioms.
- For the Fraïssé class of finite graphs, extension axioms are equivalent to extension properties of **RG**, which has limits. So, the class satisfies FO limit property.
- For the Fraïssé class of linear orders, extension axioms has limits, so it admits FO limit property.

Open problems and Further discussions

1. (**0–1 law**) When do all limits lie in $\{0, 1\}$? Is there a Fraïssé class which satisfies FO limit property but not 0-1 law?
2. (**Generic 0–1 law**) When does the limiting behavior agree with $\text{Th}(\lim(\mathcal{K}))$ (**0** if $\text{Th}(\lim(\mathcal{K})) \models \neg\phi$ and **1** if $\text{Th}(\lim(\mathcal{K})) \models \phi$)?
 - In the class of caterpillars, the FO-limit property holds, but 0-1 law fails. (This class is not Fraïssé)
 - In the class of forests of caterpillars, the FO-limit property and 0-1 law holds. This is shown by O. Heinig, T. Müller, M. Noy, and S. Taraz (2024)
 - In the class of finite linear orders, the 0-1 law holds, but generic 0-1 law fails. (With the sentence encoding density property)

References

-  R. Fraïssé, *Theory of Relations*, North-Holland, 1986.
-  W. Hodges, *A Shorter Model Theory*, Cambridge University Press, 1997.
-  K. Tent and M. Ziegler, *A Course in Model Theory*, Cambridge University Press, 2012.
-  R. Fagin, Probabilities on finite models, *Journal of Symbolic Logic* 41 (1976), 50–58.
-  P. Billingsley, *Convergence of Probability Measures*, Wiley, 1999.
-  O. Heinig, T. Müller, M. Noy, and S. Taraz, *Logical limit laws for minor-closed classes of graphs*, *Journal of Combinatorial Theory, Series B* 130 (2018), 158–206.

Thank you!