Model theoretic approaches to Szemerédi's regularity lemma

Hyoyoon Lee

Sogang University

The Fifth Korea Logic Day 2026
KAIST, Daejeon
January 14, 2026

Regularity of a graph

Definition (Graph)

In this talk, a **graph** is a structure with an irreflexive and symmetric binary relation symbol. A **vertex** is an element of the universe of a graph, and a pair of vertices with the binary relation is called an **edge**.

Regularity of a graph

Definition (Graph)

In this talk, a **graph** is a structure with an irreflexive and symmetric binary relation symbol. A **vertex** is an element of the universe of a graph, and a pair of vertices with the binary relation is called an **edge**.

Theorem (Informal Szemerédi regularity lemma)

The vertex set of every finite graph can be partitioned into a "bounded" number of parts so that the graph looks "random-like" between "most" pairs of parts of the partition.

Regularity of a graph

Definition (Graph)

In this talk, a **graph** is a structure with an irreflexive and symmetric binary relation symbol. A **vertex** is an element of the universe of a graph, and a pair of vertices with the binary relation is called an **edge**.

Theorem (Informal Szemerédi regularity lemma)

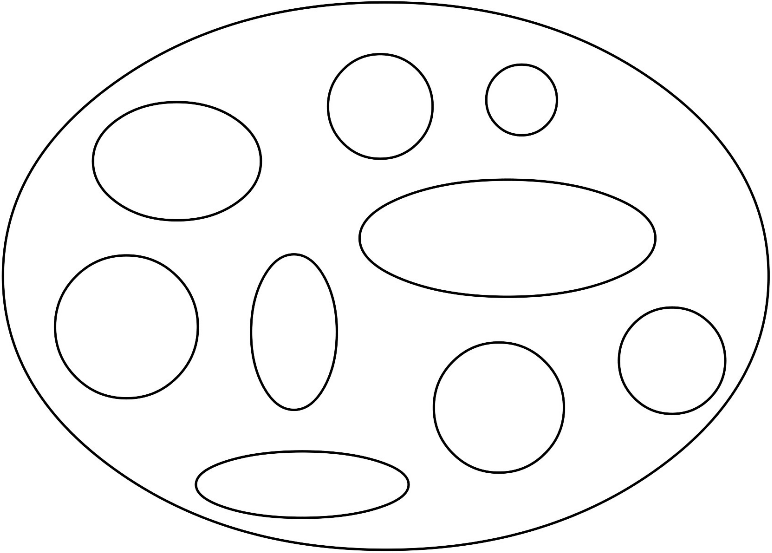
The vertex set of every finite graph can be partitioned into a "bounded" number of parts so that the graph looks "random-like" between "most" pairs of parts of the partition.

Definition (Edge density)

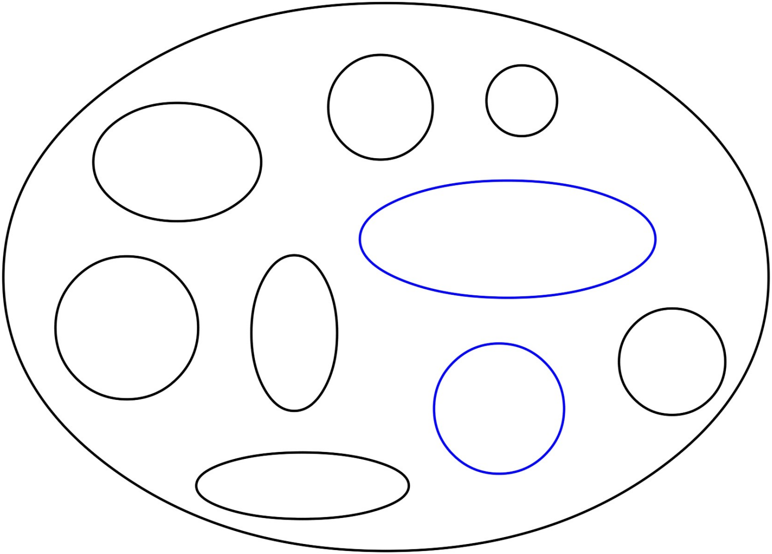
Define the **edge density** between X and Y in $G = (V, E)$ by

$$d(X, Y) := \frac{|E(X, Y)|}{|X||Y|}.$$

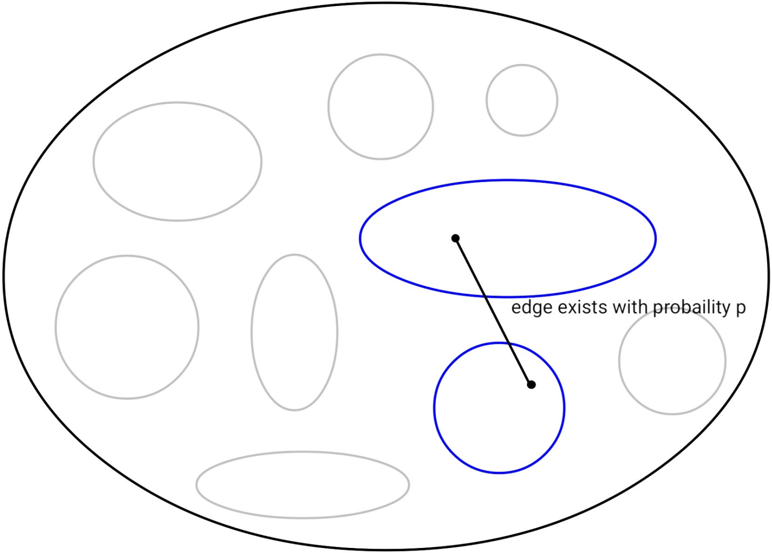
What is Random-like?



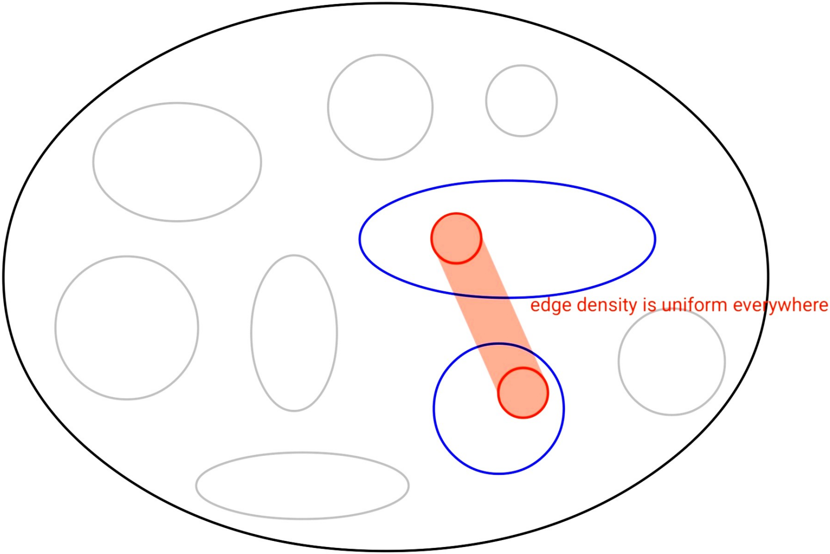
What is Random-like?



What is Random-like?



What is Random-like?



What is Random-like?

Random-likeness in terms of edge density can be reformulated as follows: Local edge density is always almost the same as the global edge density (which should be true if a graph is “truly random”).

Stable graphs

Can we (dramatically) improve Szemerédi regularity lemma?

Stable graphs

Can we (dramatically) improve Szemerédi regularity lemma?
It is impossible in general, but possible for "tame" graphs.

Stable graphs

Can we (dramatically) improve Szemerédi regularity lemma?
It is impossible in general, but possible for "tame" graphs.

Definition (half graph)

1. A k-**half graph** is a graph $V = \{a_0, \ldots a_{k-1}, b_0, \ldots, b_{k-1}\}$ satisfying:

   For any $i, j < k$, $E(a_i, b_j)$ if and only if $i < j$.

2. A graph not containing a k-half graph is called a k-**stable** graph.

Stable graphs

Can we (dramatically) improve Szemerédi regularity lemma?
It is impossible in general, but possible for "tame" graphs.

Definition (half graph)

1. A k-**half graph** is a graph $V = \{a_0, \ldots a_{k-1}, b_0, \ldots, b_{k-1}\}$ satisfying:

$$\text{For any } i, j < k, \;\; E(a_i, b_j) \text{ if and only if } i < j.$$

2. A graph not containing a k-half graph is called a k-**stable** graph.

Remark

It is a localized and finitized notion of the central notion of model theory, stability.

Stable regularity lemma

Theorem (Informal stable regularity lemma)

*The vertex set of every k-**stable** graph can be partitioned into a "bounded" number of parts (depending on k, ϵ) so that the graph looks **almost empty or complete** between **all** pairs of parts of the partition.*

Stable regularity lemma

Theorem (Informal stable regularity lemma)

*The vertex set of every k-***stable*** graph can be partitioned into a "bounded" number of parts (depending on k, ϵ) so that the graph looks* **almost empty or complete** *between* **all** *pairs of parts of the partition.*

Using the following facts, we can prove above theorem for **infinite** graphs.

Stable regularity lemma

Theorem (Informal stable regularity lemma)

*The vertex set of every k-**stable** graph can be partitioned into a "bounded" number of parts (depending on k, ϵ) so that the graph looks* **almost empty or complete** *between* **all** *pairs of parts of the partition.*

Using the following facts, we can prove above theorem for **infinite** graphs.

Theorem

For any Boolean space of finite CB-rank, any Borel probability measure is a weighted average of Dirac measures.

Stable regularity lemma

Theorem (Informal stable regularity lemma)

*The vertex set of every k-**stable** graph can be partitioned into a "bounded" number of parts (depending on k, ϵ) so that the graph looks **almost empty or complete** between **all** pairs of parts of the partition.*

Using the following facts, we can prove above theorem for **infinite** graphs.

Theorem

For any Boolean space of finite CB-rank, any Borel probability measure is a weighted average of Dirac measures.

Corollary

Let $\varphi(\bar{x}, \bar{y})$ be a stable formula. Then every Keisler φ-measure over a model is a weighted average of complete φ-types.

Stable regularity lemma

Theorem (Informal stable regularity lemma)

*The vertex set of every k-***stable*** graph can be partitioned into a "bounded" number of parts (depending on k, ϵ) so that the graph looks* **almost empty or complete** *between* **all** *pairs of parts of the partition.*

Using the following facts, we can prove above theorem for **infinite** graphs.

Theorem

For any Boolean space of finite CB-rank, any Borel probability measure is a weighted average of Dirac measures.

Corollary

Let $\varphi(\bar{x}, \bar{y})$ be a stable formula. Then every Keisler φ-measure over a model is a weighted average of complete φ-types.

Question: Can we derive the theorem also for finite graphs?

Definition (Filter)

Let $\mathcal{F}$ be a family of subsets of ω. $\mathcal{F}$ is called a **filter** on ω if

- $\emptyset \notin \mathcal{F}$,
- if $A, B \in \mathcal{F}$, then $A \cap B \in \mathcal{F}$, and
- if $A \in \mathcal{F}$ and $A \subseteq B \subseteq \omega$, then $B \in \mathcal{F}$.

ω can be replaced with any set, but we will consider only ω.

Ultrafilter

Definition (Filter)

Let $\mathcal{F}$ be a family of subsets of ω. $\mathcal{F}$ is called a **filter** on ω if

- $\emptyset \notin \mathcal{F}$,
- if $A, B \in \mathcal{F}$, then $A \cap B \in \mathcal{F}$, and
- if $A \in \mathcal{F}$ and $A \subseteq B \subseteq \omega$, then $B \in \mathcal{F}$.

ω can be replaced with any set, but we will consider only ω.

Definition (Ultrafilter)

If $\mathcal{F}$ is a filter and A or $\omega \setminus A$ belongs to $\mathcal{F}$ for any $A \subseteq \omega$, then it is called an **ultrafilter** on ω.

Ultraproduct

Notation

1. For any finite tuple $\bar{a} = a_0 a_1 \cdots a_{n-1} \in (\prod_{i\in\omega} M_i)^n$, $\bar{a}[i]$ is the projection to the i-th coordinate of ω:
$$\bar{a}[i] = a_0[i]a_1[i]\cdots a_{n-1}[i] \in (M_i)^n.$$
2. For any finite tuples $\bar{a}, \bar{b} \in (\prod_{i\in\omega} M_i)^n$, $\bar{a} \sim_{\mathcal{U}} \bar{b}$ if $\{i : \bar{a}[i] = \bar{b}[i]\} \in \mathcal{U}$.

Ultraproduct

Notation

1. For any finite tuple $\bar{a} = a_0 a_1 \cdots a_{n-1} \in (\prod_{i\in\omega} M_i)^n$, $\bar{a}[i]$ is the projection to the i-th coordinate of ω:
$$\bar{a}[i] = a_0[i] a_1[i] \cdots a_{n-1}[i] \in (M_i)^n.$$
2. For any finite tuples $\bar{a}, \bar{b} \in (\prod_{i\in\omega} M_i)^n$, $\bar{a} \sim_{\mathcal{U}} \bar{b}$ if $\{i : \bar{a}[i] = \bar{b}[i]\} \in \mathcal{U}$.

Let $(M_i)_{i\in\omega}$ be a sequence of $\mathcal{L}$-structures. The **ultraproduct** of $\{M_i : i \in \omega\}$ over a ultrafilter $\mathcal{U}$, denoted by $\mathfrak{M} = \prod_{i\in\omega} M_i/\mathcal{U}$, is defined to be an $\mathcal{L}$-structure as follows:

Ultraproduct

Notation

1. For any finite tuple $\bar{a} = a_0 a_1 \cdots a_{n-1} \in (\prod_{i\in\omega} M_i)^n$, $\bar{a}[i]$ is the projection to the i-th coordinate of ω:
$$\bar{a}[i] = a_0[i]a_1[i] \cdots a_{n-1}[i] \in (M_i)^n.$$
2. For any finite tuples $\bar{a}, \bar{b} \in (\prod_{i\in\omega} M_i)^n$, $\bar{a} \sim_{\mathcal{U}} \bar{b}$ if $\{i : \bar{a}[i] = \bar{b}[i]\} \in \mathcal{U}$.

Let $(M_i)_{i\in\omega}$ be a sequence of $\mathcal{L}$-structures. The **ultraproduct** of $\{M_i : i \in \omega\}$ over a ultrafilter $\mathcal{U}$, denoted by $\mathfrak{M} = \prod_{i\in\omega} M_i/\mathcal{U}$, is defined to be an $\mathcal{L}$-structure as follows:

- The universe of $\mathfrak{M} = \prod_{i\in\omega} M_i/\mathcal{U}$ is $\{a/\sim_{\mathcal{U}} : a \in \prod_{i\in\omega} M_i\}$.

Ultraproduct

Notation

1. For any finite tuple $\bar{a} = a_0 a_1 \cdots a_{n-1} \in (\prod_{i\in\omega} M_i)^n$, $\bar{a}[i]$ is the projection to the i-th coordinate of ω:
$$\bar{a}[i] = a_0[i]a_1[i]\cdots a_{n-1}[i] \in (M_i)^n.$$
2. For any finite tuples $\bar{a}, \bar{b} \in (\prod_{i\in\omega} M_i)^n$, $\bar{a} \sim_{\mathcal{U}} \bar{b}$ if $\{i : \bar{a}[i] = \bar{b}[i]\} \in \mathcal{U}$.

Let $(M_i)_{i\in\omega}$ be a sequence of $\mathcal{L}$-structures. The **ultraproduct** of $\{M_i : i \in \omega\}$ over a ultrafilter $\mathcal{U}$, denoted by $\mathfrak{M} = \prod_{i\in\omega} M_i/\mathcal{U}$, is defined to be an $\mathcal{L}$-structure as follows:

- The universe of $\mathfrak{M} = \prod_{i\in\omega} M_i/\mathcal{U}$ is $\{a/\sim_{\mathcal{U}} : a \in \prod_{i\in\omega} M_i\}$.
- For any relation $R \in \mathcal{L}$, $\mathfrak{M} \models R(\bar{a})$ if and only if $\{i \in \omega : M_i \models R(\bar{a}[i])\} \in \mathcal{U}$.

Ultraproduct

Notation

1. For any finite tuple $\bar{a} = a_0 a_1 \cdots a_{n-1} \in (\prod_{i\in\omega} M_i)^n$, $\bar{a}[i]$ is the projection to the i-th coordinate of ω:
$$\bar{a}[i] = a_0[i] a_1[i] \cdots a_{n-1}[i] \in (M_i)^n.$$
2. For any finite tuples $\bar{a}, \bar{b} \in (\prod_{i\in\omega} M_i)^n$, $\bar{a} \sim_{\mathcal{U}} \bar{b}$ if $\{i : \bar{a}[i] = \bar{b}[i]\} \in \mathcal{U}$.

Let $(M_i)_{i\in\omega}$ be a sequence of $\mathcal{L}$-structures. The **ultraproduct** of $\{M_i : i \in \omega\}$ over a ultrafilter $\mathcal{U}$, denoted by $\mathfrak{M} = \prod_{i\in\omega} M_i/\mathcal{U}$, is defined to be an $\mathcal{L}$-structure as follows:

- The universe of $\mathfrak{M} = \prod_{i\in\omega} M_i/\mathcal{U}$ is $\{a/\sim_{\mathcal{U}} : a \in \prod_{i\in\omega} M_i\}$.
- For any relation $R \in \mathcal{L}$, $\mathfrak{M} \models R(\bar{a})$ if and only if $\{i \in \omega : M_i \models R(\bar{a}[i])\} \in \mathcal{U}$.
- For any function $F \in \mathcal{L}$, $\mathfrak{M} \models F(\bar{a}) = b$ if and only if $\{i \in \omega : M_i \models F(\bar{a}[i]) = b\} \in \mathcal{U}$.

Łoś Theorem

Theorem (Łoś Theorem)

Let $\{M_i : i \in \omega\}$ *be a sequence of* $\mathcal{L}$*-structures and* $\mathcal{U}$ *be an ultrafilter on* ω*. Then for any* $\mathcal{L}$*-formula* $\varphi(\bar{x})$ *and any finite tuple* $\bar{a}$ *in* $\mathfrak{M}$*,*

$$\mathfrak{M} \models \varphi(\bar{a}) \text{ if and only if } \{i \in \omega : M_i \models \varphi(\bar{a}[i])\} \in \mathcal{U}.$$

Łoś Theorem

Theorem (Łoś Theorem)

Let $\{M_i : i \in \omega\}$ *be a sequence of* $\mathcal{L}$*-structures and* $\mathcal{U}$ *be an ultrafilter on* ω*. Then for any* $\mathcal{L}$*-formula* $\varphi(\bar{x})$ *and any finite tuple* $\bar{a}$ *in* $\mathfrak{M}$*,*

$$\mathfrak{M} \models \varphi(\bar{a}) \text{ if and only if } \{i \in \omega : M_i \models \varphi(\bar{a}[i])\} \in \mathcal{U}.$$

An ultrafilter $\mathcal{U}$ is called **principal** if there is $A \subseteq \omega$ such that $\mathcal{U} = \{B \subseteq \omega : A \subseteq B\}$.

Łoś Theorem

Theorem (Łoś Theorem)

Let $\{M_i : i \in \omega\}$ *be a sequence of* $\mathcal{L}$*-structures and* $\mathcal{U}$ *be an ultrafilter on* ω*. Then for any* $\mathcal{L}$*-formula* $\varphi(\bar{x})$ *and any finite tuple* $\bar{a}$ *in* $\mathfrak{M}$*,*

$$\mathfrak{M} \models \varphi(\bar{a}) \text{ if and only if } \{i \in \omega : M_i \models \varphi(\bar{a}[i])\} \in \mathcal{U}.$$

An ultrafilter $\mathcal{U}$ is called **principal** if there is $A \subseteq \omega$ such that $\mathcal{U} = \{B \subseteq \omega : A \subseteq B\}$.

Remark

- If $\mathcal{U}$ is nonprincipal, then $\mathcal{U}$ contains only infinite subsets of ω.

Łoś Theorem

Theorem (Łoś Theorem)

Let $\{M_i : i \in \omega\}$ be a sequence of $\mathcal{L}$-structures and $\mathcal{U}$ be an ultrafilter on ω. Then for any $\mathcal{L}$-formula $\varphi(\bar{x})$ and any finite tuple $\bar{a}$ in $\mathfrak{M}$,

$$\mathfrak{M} \models \varphi(\bar{a}) \text{ if and only if } \{i \in \omega : M_i \models \varphi(\bar{a}[i])\} \in \mathcal{U}.$$

An ultrafilter $\mathcal{U}$ is called **principal** if there is $A \subseteq \omega$ such that $\mathcal{U} = \{B \subseteq \omega : A \subseteq B\}$.

Remark

- If $\mathcal{U}$ is nonprincipal, then $\mathcal{U}$ contains only infinite subsets of ω.
- Thus intuitively, if $\mathcal{U}$ is nonprincipal, then $\{i \in \omega : M_i \models \varphi(\bar{a}[i])\} \in \mathcal{U}$ means that $M_i \models \varphi(\bar{a})$ for **almost every** $i \in \omega$.

Łoś Theorem

Theorem (Łoś Theorem)

Let $\{M_i : i \in \omega\}$ *be a sequence of* $\mathcal{L}$*-structures and* $\mathcal{U}$ *be an ultrafilter on* ω*. Then for any* $\mathcal{L}$*-formula* $\varphi(\bar{x})$ *and any finite tuple* $\bar{a}$ *in* $\mathfrak{M}$*,*

$$\mathfrak{M} \models \varphi(\bar{a}) \text{ if and only if } \{i \in \omega : M_i \models \varphi(\bar{a}[i])\} \in \mathcal{U}.$$

An ultrafilter $\mathcal{U}$ is called **principal** if there is $A \subseteq \omega$ such that $\mathcal{U} = \{B \subseteq \omega : A \subseteq B\}$.

Remark

- If $\mathcal{U}$ is nonprincipal, then $\mathcal{U}$ contains only infinite subsets of ω.
- Thus intuitively, if $\mathcal{U}$ is nonprincipal, then $\{i \in \omega : M_i \models \varphi(\bar{a}[i])\} \in \mathcal{U}$ means that $M_i \models \varphi(\bar{a})$ for **almost every** $i \in \omega$.
- Ultraproducts can give good notions of limits.

Moving from infinite to finite

Structure of the proof using ultraproducts, in general.

1. Suppose that the desired theorem for finite graphs is not true, then we can get a sequence of finite graphs that witness the failure of the theorem.

Moving from infinite to finite

Structure of the proof using ultraproducts, in general.

1. Suppose that the desired theorem for finite graphs is not true, then we can get a sequence of finite graphs that witness the failure of the theorem.
2. Take an ultraproduct $\mathfrak{M}$ of this sequence of graphs. Then this graphs is an **infinite** graph sharing first-order properties (e.g. k-stable) of the given finite graphs.

Moving from infinite to finite

Structure of the proof using ultraproducts, in general.

1. Suppose that the desired theorem for finite graphs is not true, then we can get a sequence of finite graphs that witness the failure of the theorem.
2. Take an ultraproduct $\mathfrak{M}$ of this sequence of graphs. Then this graphs is an **infinite** graph sharing first-order properties (e.g. k-stable) of the given finite graphs.
3. Apply a "useful fact" on the infinite or continuous objects to get a useful fact on the definable sets of the ultraproduct.

Moving from infinite to finite

Structure of the proof using ultraproducts, in general.

1. Suppose that the desired theorem for finite graphs is not true, then we can get a sequence of finite graphs that witness the failure of the theorem.
2. Take an ultraproduct $\mathfrak{M}$ of this sequence of graphs. Then this graphs is an **infinite** graph sharing first-order properties (e.g. k-stable) of the given finite graphs.
3. Apply a "useful fact" on the infinite or continuous objects to get a useful fact on the definable sets of the ultraproduct.
4. Use Łoś Theorem to show that this useful fact on the definable sets also holds for almost every finite graph in the sequence.

Moving from infinite to finite

Structure of the proof using ultraproducts, in general.

1. Suppose that the desired theorem for finite graphs is not true, then we can get a sequence of finite graphs that witness the failure of the theorem.
2. Take an ultraproduct $\mathfrak{M}$ of this sequence of graphs. Then this graphs is an **infinite** graph sharing first-order properties (e.g. k-stable) of the given finite graphs.
3. Apply a "useful fact" on the infinite or continuous objects to get a useful fact on the definable sets of the ultraproduct.
4. Use Łoś Theorem to show that this useful fact on the definable sets also holds for almost every finite graph in the sequence.
5. Point out that this useful fact is contradictory to the failure of the theorem for finite graphs. □

Concluding remarks

In fact, we can say more in the stable regularity lemma.

Theorem (Informal stable regularity lemma)

*The vertex set of every k-***stable** *graph can be partitioned into a $f(k)\cdot$***polynomial**(ϵ) *number of parts so that the graph looks* **almost empty or complete** *between* **all** *pairs of parts of the partition.*

Concluding remarks

In fact, we can say more in the stable regularity lemma.

Theorem (Informal stable regularity lemma)

*The vertex set of every k-***stable** *graph can be partitioned into a $f(k)\cdot$***polynomial**(ϵ) *number of parts so that the graph looks* **almost empty or complete** *between* **all** *pairs of parts of the partition.*

To prove it, we need to finitize the proof of the theorem for infinite graphs. If we use ultraproducts, then we essentially use compactness, so we can't keep some numerical information.

Concluding remarks

In fact, we can say more in the stable regularity lemma.

Theorem (Informal stable regularity lemma)

*The vertex set of every k-**stable** graph can be partitioned into a $f(k)\cdot$**polynomial**(ϵ) number of parts so that the graph looks **almost empty or complete** between **all** pairs of parts of the partition.*

To prove it, we need to finitize the proof of the theorem for infinite graphs.
If we use ultraproducts, then we essentially use compactness, so we can't keep some numerical information.
So, is ultraproduct useless in this context?

Concluding remarks

In fact, we can say more in the stable regularity lemma.

Theorem (Informal stable regularity lemma)

*The vertex set of every k-***stable** *graph can be partitioned into a $f(k)$·***polynomial**(ϵ) *number of parts so that the graph looks* **almost empty or complete** *between* **all** *pairs of parts of the partition.*

To prove it, we need to finitize the proof of the theorem for infinite graphs. If we use ultraproducts, then we essentially use compactness, so we can't keep some numerical information.
So, is ultraproduct useless in this context? Never, for the following reasons.

Concluding remarks

In fact, we can say more in the stable regularity lemma.

Theorem (Informal stable regularity lemma)

*The vertex set of every k-**stable** graph can be partitioned into a $f(k)\cdot$**polynomial**(ϵ) number of parts so that the graph looks **almost empty or complete** between **all** pairs of parts of the partition.*

To prove it, we need to finitize the proof of the theorem for infinite graphs. If we use ultraproducts, then we essentially use compactness, so we can't keep some numerical information.

So, is ultraproduct useless in this context? Never, for the following reasons.

- It can build a bridge between finite and infinite structures; theorems on finite suggest that there is an analogue for infinite, and vice versa.

Concluding remarks

In fact, we can say more in the stable regularity lemma.

Theorem (Informal stable regularity lemma)

*The vertex set of every k-**stable** graph can be partitioned into a $f(k)\cdot$**polynomial**(ϵ) number of parts so that the graph looks **almost empty or complete** between **all** pairs of parts of the partition.*

To prove it, we need to finitize the proof of the theorem for infinite graphs. If we use ultraproducts, then we essentially use compactness, so we can't keep some numerical information.

So, is ultraproduct useless in this context? Never, for the following reasons.

- It can build a bridge between finite and infinite structures; theorems on finite suggest that there is an analogue for infinite, and vice versa.
- When we are interested only on the existence of some number, then the method of ultraproduct is not defective.

Concluding remarks

In fact, we can say more in the stable regularity lemma.

Theorem (Informal stable regularity lemma)

*The vertex set of every k-**stable** graph can be partitioned into a $f(k)\cdot$**polynomial**(ϵ) number of parts so that the graph looks **almost empty or complete** between **all** pairs of parts of the partition.*

To prove it, we need to finitize the proof of the theorem for infinite graphs. If we use ultraproducts, then we essentially use compactness, so we can't keep some numerical information.

So, is ultraproduct useless in this context? Never, for the following reasons.

- It can build a bridge between finite and infinite structures; theorems on finite suggest that there is an analogue for infinite, and vice versa.
- When we are interested only on the existence of some number, then the method of ultraproduct is not defective.
- Using ultraproducts, we can forget about numeric computations and reveal that there is more "theoretic" reason why the theorem holds.

References

[MS14] M. MALLIARIS and S. SHELAH. "REGULARITY LEMMAS FOR STABLE GRAPHS". In: *Transactions of the American Mathematical Society* 366.3 (2014), pp. 1551–1585.

[Pil24] Anand Pillay. *Topics in Model Theory*. WORLD SCIENTIFIC, 2024.

[Zha23] Yufei Zhao. *Graph Theory and Additive Combinatorics: Exploring Structure and Randomness*. Cambridge University Press, 2023.

Happy Korea Logic Day!