Introduction to continuous logic

Stability and examples

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Definition 1

There are 2 different approaches to stability in metric structures.

- We say T is *λ-stable with respect to the discrete metric* if every $\mathcal{M} \models T$ and every $A \subseteq M$ of cardinality $\leq \lambda$, the set $S_1(T_A)$ has cardinality $\leq \lambda$.
- We say that T is *stable with respect to the discrete metric* if T is λ-stable with respect to the discrete metric for some λ.

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- We say that T is λ-*stable* if for every $\mathcal{M} \models T$ and $A \subseteq M$ of cardinality $\leq \lambda$, there is a subset of $S_1(T_A)$ of cardinality $\leq \lambda$ that is dense in $S_1(T_A)$ with respect to d-metric.
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Fact A theory T is stable iff T is stable with respect to the discrete metric.

Let $\mathcal{M}$ be a κ-universal domain for T and let A, B, C be small subsets of M. A ternary relation $\downarrow$ is called a *stable independence relation* if it satisfies the following:

1. Invariance under automorphisms of $\mathcal{M}$.
2. Symmetry: $A \downarrow_C B \iff B \downarrow_C A$.
3. Transitivity: $A \downarrow_C (B \cup D)$ if and only if $A \downarrow_C B$ and $A \downarrow_{B \cup C} D$.
4. Finite character: $A \downarrow_C B$ if and only if $a \downarrow_C B$ for all finite tuples a from A.
5. Extension: for all A, B, C, there is A' such that $A' \downarrow_C B$ and $\mathrm{tp}(A/C) = \mathrm{tp}(A'/C)$.
6. Local character: for every finite tuple a, there is $B' \subseteq B$ of cardinality $\leq |T|$ such that $a \downarrow_{B'} B$.
7. Stationarity of types: if $\mathrm{tp}(A/M) = \mathrm{tp}(A'/M)$, $A \downarrow_M B$, and $A' \downarrow_M B$, then we have $\mathrm{tp}(A/B \cup M) = \mathrm{tp}(A'/B \cup M)$.

Theorem (BBHU)

Let $\mathcal{M}$ be a κ-universal domain for T. If T is stable, then there is precisely one stable independence relation on $\mathcal{M}$. Moreover, if there exists a stable independence relation $\mathop{\smile\!\!\!\!\!\!|}^{*}$ on triples of small subsets of M, then T is stable.

Examples

- The theory of infinite dimensional Hilbert space is ω-stable, and the stable independence relation is orthogonality.
- The theory of atomless probability algebras is ω-stable, and the stable independence relation is probabilistic independence. (coming soon!)

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Signature of probability algebras

The signature is $L_{Pr} = \{0, 1, \cap, \cup, \complement, \mu\}$, where

- 0,1 are constant symbols
- $\cap, \cup$ are binary function symbols
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Note that

- $\complement$ and μ are 1-Lipschitz,
- $\cap$ and $\cup$ are 2-Lipschitz.

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Formulas

- $x \cap y, (x\complement) \cup z$ are terms;
- $\mu(x \triangle y), d(x, y), |\mu(x) - \mu(y)|$ are formulas;
- $\sup_x \sup_y d(x, y) \dot{-} 1, \mu(0)$ are sentences;
- $\sup_x \sup_y (d(x, y) \dot{-} 1) = 0, \mu(0) = 0$ are closed conditions.

Example

- Let (X, Σ, μ) be a probability space and $I \subset \Sigma$ the μ-null sets. Then $\mathcal{B} = \Sigma / I$ is called the probability measured algebra.
- Define $d(a, b) = \mu(a \triangle b)$ for all $a, b \in \mathcal{B}$.

Check $M = (\mathcal{B}, 0, 1, \cap, \cup, \complement, \mu, d)$ is an L_{Pr}-structure.

- Interpretation of functions and predicates is natural here.

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Axioms of probability algebras

Let Pr denote the L_{Pr}-theory consists of the following axioms:

1. boolean algebra axioms (next slides)
2. measure axioms
   - $\mu(\mathbf{0}) = 0$
   - $\mu(\mathbf{1}) = 1$
   - $\sup_x \sup_y \left| \frac{\mu(x \cup y) + \mu(x \cap y)}{2} - \frac{\mu(x) + \mu(y)}{2} \right| = 0$
     i.e., $\forall x \forall y \mu(x \cup y) + \mu(x \cap y) = \mu(x) + \mu(y)$
3. metric axiom: $\sup_x \sup_y |d(x, y) - \mu(x \triangle y)| = 0$,
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Boolean algebra axioms

- $\sup_x \sup_y \big(d(x \cup y, y \cup x)\big) = 0$
- $\sup_x \sup_y \big(d(x \cap y, y \cap x)\big) = 0$
- $\sup_x \sup_y \sup_z d\big(x \cup (y \cup z), (x \cup y) \cup z\big) = 0$
- $\sup_x \sup_y \sup_z d\big(x \cap (y \cap z), (x \cap y) \cap z\big) = 0$
- $\sup_x \sup_y d\big(x \cup (x \cap y), x\big) = 0$
- $\sup_x \sup_y d\big(x \cap (x \cup y), x\big) = 0$
- $\sup_x \sup_y \sup_z d\big(x \cup (y \cap z), (x \cup y) \cap (x \cup z)\big) = 0$
- $\sup_x \sup_y \sup_z d\big(x \cap (y \cup z), (x \cap y) \cup (x \cap z)\big) = 0$
- $\sup_x d(x \cup x^{\complement}, \mathbf{1}) = 0$
- $\sup_x d(x \cap x^{\complement}, \mathbf{0}) = 0$

Proposition (Berenstein and Henson)

$\mathcal{M} \models \mathrm{Pr}$ *iff* $\mathcal{M}$ *is isomorphic to a probability measured algebra of a probability space.*

Atomless probability algebras

The theory APr is Pr together the following atomless axiom,

$$\sup_x \inf_y |\mu(x \cap y) - \mu(x \cap y^{\complement})| = 0$$

Then $\mathcal{M} \models \text{APr} \Rightarrow \mathcal{M}$ is atomless.

Theorem (Ben Yaacov)

Let $\mathcal{M}$ be a model of APr. *Then:*

1. *The theory* APr *is separably categorical and complete.*
2. *The universal part of* APr *is* Pr, *and* APr *is the model companion of* Pr.
3. *The theory* APr *admits quantifier elimination.*
4. *Assume that $\mathcal{M}$ is the probability structure associated to a probability space $(\Omega, \mathcal{F}, \mu)$. Let $C \subseteq M$ and let $\mathcal{C}$ be the σ-algebra σ-generated by the events in C. For any two n-tuples $a, b \in M^n$, we have $\mathrm{tp}(a/C) = \mathrm{tp}(b/C)$ if and only if $\mathbb{P}(a_1^{\epsilon_1} \cap \cdots \cap a_n^{\epsilon_n} | \mathcal{C}) = \mathbb{P}(b_1^{\epsilon_1} \cap \cdots \cap b_n^{\epsilon_n} | \mathcal{C})$ a.s. for all $\epsilon_1, \cdots, \epsilon_n \in \{-1, 1\}$.*

Theorem (Ben Yaacov)

Let $\mathcal{M}$ be a model of APr*. Then, the theory* APr *is ω-stable and its independence relation coincides with the probabilistic independence relation. That is, if A, B, C are subsets of a model of* APr*, then $A \mathop{\smile\!\!\!\!\!\!\mid}_C B$ iff*

$$\mathbb{P}(a \cap b \mid \sigma(C)) = \mathbb{P}(a \mid \sigma(C))\mathbb{P}(b \mid \sigma(C)) \quad a.s.,$$

for all $a \in \sigma(A)$ and $b \in \sigma(B)$.

Theorem (S.)

Let $\mathcal{M} \models \mathrm{APr}$ and suppose that $\mathcal{M}$ is associated to an atomless probability space Ω. Then the following are equivalent:

1. *$\mathcal{M}$ is strongly $||M||$-homogeneous.*
2. *$\mathcal{M}$ is strongly ω-homogeneous.*

Hoover-Keisler Saturation

- Let Ω and Γ be probability spaces and let $\mathcal{X}$ be a Polish metric space. Let $x, y : \Gamma \to \mathcal{X}$ be random variables. The probability space Ω is said to have the **saturation property for** $dist(x, y)$ if for every random variable $x' : \Omega \to \mathcal{X}$ with $dist(x) = dist(x')$, there is a random variable $y' : \Omega \to \mathcal{X}$ such that $dist(x, y) = dist(x', y')$.
- A probability space Ω is said to be **Hoover-Keisler saturated** if for all two random variables $x, y : \Gamma \to \mathcal{X}$, where Γ is an arbitrary probability space and $\mathcal{X}$ is an arbitrary Polish metric space, Ω has the saturation property for $dist(x, y)$.

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Hoover and Keisler's work

Hoover and Keisler showed that many properties, such as existence of solutions of stochastic integral equations, regularity properties for distributions of correspondences, and the existence of pure strategy equilibria in games with many players, are not realized in the standard Lebesgue space, but are realized in Hoover-Keisler saturated probability spaces.

Later, I will show that Hoover-Keisler saturated probability spaces are $\aleph_1$-saturated probability spaces.

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Later, I will show that Hoover-Keisler saturated probability spaces are $\aleph_1$-saturated probability spaces.

Theorem (S.)

Let $\mathcal{M} \models \mathrm{APr}$ *and suppose that* $\mathcal{M}$ *is associated to an atomless probability space* Ω*. For every infinite cardinal* κ*, the following are equivalent:*

1. $\mathcal{M}$ *is* κ*-saturated.*
2. Ω *is* κ*-atomless.*

Corollary (S.)

A probability space Ω is atomless if and only if the probability measured algebra of Ω is $\aleph_0$-saturated.

Corollary (S.)

An atomless probability space is Hoover-Keisler saturated if and only if it is $\aleph_1$-saturated.

- In model theory, the theory of atomless boolean algebras is among the wildest theories.
- In continuous model theory, the theory of atomless probability algebras is ω-stable.
- The inclusion relation $\subset$ gives the order property in atomless boolean algebras.
- Why $\subset$ fails the order property in atomless probability algebras?

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The order property

- Let $\varphi(x, y)$ be an formula with $2n$ free variables, and let $0 \leq \epsilon < \frac{1}{2}$. Define the relation $\prec_{\varphi,\epsilon}$ by

$$a \prec_{\varphi,\epsilon} b \textit{ if } \varphi(a, b) \leq \epsilon \textit{ and } \varphi(b, a) \geq 1 - \epsilon,$$

  where $a, b \in M^n$.
- A *φ-ϵ-chain of length k in M* is a sequence of n-tuples of length k, $(a_i)_{1\leq i\leq k}$, such that $a_i \prec_{\varphi,\epsilon} a_j$ iff $i < j$.
- A theory T has *the order property* if there exists a formula $\varphi(x, y)$ such that for every $\epsilon > 0$, there exists $\mathcal{M} \models T$ such that $\mathcal{M}$ has arbitrarily long finite φ-ϵ-chains.

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APr does not have the order property

The formula $x \subset y$ does not have the order property in atomless probability algebras, because the probability measure is finite, and thus ϵ-chain can not be arbitrarily long.

NIP

- (Shattering) Let X, Y be sets, $f\colon X \times Y \to [0,1]$, $r \in (0,1)$, and $\epsilon > 0$. Let $A \subseteq X$. We say that f (r,ϵ)-*shatters* A if for every $C \subseteq A$, there exists some $b_C \in Y$ such that

$$\{a \in X \mid f(a,b_C) \leq r\} \cap A = C,$$

and

$$\{a \in X \mid f(a,b_C) \geq r+\epsilon\} \cap A = A \backslash C.$$

- Let X, Y be sets and $f\colon X \times Y \to [0,1]$. We say that f is (r,ϵ)-*independent* if for every $n \in \mathbb{N}$, there exists $A \subseteq X$ where $|A| > n$ and f (r,ϵ)-shatters A.
- (NIP) We say that f is *independent* if there exists some $r \in (0,1)$ and $\epsilon > 0$ such that f is (r,ϵ)-independent. We say that f has *NIP* if f is not independent.

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A theory T has *NIP* if for every formula φ and for every model $\mathcal{M} \models T$, $\varphi^{\mathcal{M}}$ has NIP.

L_{RV}-structures

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Example

Let $M = L^1\big([0,1],[0,1]\big)$. For all $f(x), g(x) \in M$, we have

- $\neg f(x) = 1 - f(x)$
- $\frac{1}{2}(f(x)) = f(x)/2$
- $f(x) \dot{-} g(x) = \max\{f(x) - g(x), 0\}$
- $I(f(x)) = \int_0^1 f(x)dx$

The theory of atomless random variable structures: ARV

- For a probability space $(\Omega, \mathcal{F}, \mu)$, the L_{RV}-structure $\mathcal{M} = (L^1(\mu, [0,1]), 0, 1, \neg, \frac{1}{2}, \dot{-}, I)$ is called a **random variable structure.**
- A probability space $(\Omega, \mathcal{F}, \mu)$ is **atomless** if for $B \in \mathcal{F}$ with $\mu(B) > 0$ there is $A \in \mathcal{F}$ such that $A \subset B$ and $0 < \mu(A) < \mu(B)$.
- Let ARV denote the theory of the class of all atomless random variable structures:

$$\{L^1((\Omega, \mathcal{F}, \mu), [0,1]) \mid \Omega \text{ is atomless}\}$$

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Proposition (Ben Yaacov)

$\mathcal{M} \models$ ARV *if and only if there is an atomless probability space* $(\Omega, \mathcal{F}, \mu)$ *such that* $\mathcal{M} \cong L^1(\mu, [0, 1])$.

Properties of ARV

Theorem (Ben Yaacov)

- ARV *is complete and separably categorical.*
- *The universal part of* ARV *is* RV *and* ARV *is the model companion of* RV*.*
- ARV *admits quantifier elimination.*
- ARV *is* ω*-stable and independence coinciding with probabilistic independence.*
- *Let* A *be a subset of a model of* ARV*. Then* $tp(f/A) = tp(g/A)$ *iff* f *and* g *have the same joint conditional distributions over* $\sigma(A)$*, the* σ*-algebra of measurable sets generated by random variables in* A*.*

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Theorem (Keisler)

If $\mathcal{M} \models \mathrm{ARV}$ is separable, then $\mathcal{M}$ is not $\aleph_0$-saturated.

Theorem (S.)

Let $\mathcal{M} \models \mathrm{ARV}$. Suppose that $M = L^1(\mathcal{B}, [0,1])$ for some $\mathcal{B} \models \mathrm{APr}$. Then $\mathcal{M} \models \mathrm{ARV}$ is κ-saturated if and only if $\mathcal{B} \models \mathrm{APr}$ is κ-saturated, for every uncountable cardinal κ.

Proposition (S.)

The following are equivalent:

1. Ω *is Hoover-Keisler saturated.*
2. Ω *is* $\aleph_1$*-saturated.*
3. $L^1(\Omega, [0,1])$ *as a model of* ARV *is* $\aleph_1$*-saturated.*
4. $L^1(\Omega, [0,1])$ *as a model of* ARV *is* $\aleph_0$*-saturated.*
5. *For all elements* a, b, c *in a model* $\mathcal{M}$ *of* ARV *with* $tp(a) = tp(b)$*, there exists* $d \in M$ *such that* $tp(a, c) = tp(b, d)$*. (2-saturated)*

Conditional distributions

Let $(\Omega, \mathcal{B}, m)$ be a probability space and $\mathcal{A}$ a probability subalgebra of $\mathcal{B}$. An $L^1\big((\Omega, \mathcal{A}, m), [0,1]\big)$-valued Borel probability measure μ on $\mathbb{R}^n$ is called a **conditional distribution over** $\mathcal{A}$ if it satisfies:

1. $\mu(B) \geq 0$ *a.s.* for Borel $B \subseteq \mathbb{R}^n$
2. $\mu(\mathbb{R}^n) = 1$ *a.s.*
3. $\mu(\cup_{i=1}^{\infty} B_i) = \sum_{i=1}^{\infty} \mu(B_i)$ *a.s.* for disjoint Borel $B_1, B_2, \cdots, \subseteq \mathbb{R}^n$

Let $\mathfrak{D}_X(\mathcal{A})$ denote the space of all conditional distributions over $\mathcal{A}$ which, as measures, are supported by $X \subseteq \mathbb{R}^n$.

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Let $\mathfrak{D}_X(\mathcal{A})$ denote the space of all conditional distributions over $\mathcal{A}$ which, as measures, are supported by $X \subseteq \mathbb{R}^n$.

An Example

Let $f\colon \Omega \to \mathbb{R}^n$. Then f determines a conditional distribution over $\mathcal{A}$, denoted $dist(f \mid \mathcal{A})$. For Borel $B \subseteq \mathbb{R}^n$,

$$dist(f \mid \mathcal{A})(B) := \mathbb{E}(\mathbf{1}_{\{f \in B\}} \mid \mathcal{A}).$$

Correspondence

Theorem (Ben Yaacov)

Let f be an n-tuple in a model M of ARV and A be a subset of M. Let $\mathcal{A}$ denote $\sigma(A)$, the σ-algebra of measurable sets generated by random variables in A. Then the joint conditional distribution $dist(f \mid \mathcal{A})$ only depends on $tp(f/A)$. Moreover, the mapping

$$\begin{aligned} \zeta\colon S_n(A) &\to \mathfrak{D}_{[0,1]^n}(\mathcal{A}) \\ tp(f/A) &\mapsto dist(f \mid \mathcal{A}) \end{aligned}$$

is a homeomorphism between $S_n(A)$ equipped with the logic topology and $\mathfrak{D}_{[0,1]^n}(\mathcal{A})$ equipped with the topology of weak convergence.

The formula for $n = 1$

Theorem (S.)

Let κ be an uncountable cardinal. Let $M \models \mathrm{ARV}$ be a κ-saturated model of the form $M = L^1(m, [0,1])$, where $(\Omega, \mathcal{F}, m)$ is an atomless probability space. Suppose $C \subseteq M$ is small. Let $\mathcal{C}$ be the σ-algebra of measurable sets generated by random variables in C. For all $a, b \in M$,

$$d^*(tp(a/C), tp(b/C)) = \int_0^1 \left\| \mathbb{P}(a > t|\mathcal{C}) - \mathbb{P}(b > t|\mathcal{C}) \right\|_1 dt$$

where $\| \cdot \|_1$ is the L^1-norm.

But for general n, the explicit formula is more complicated and less elegant. I used results in optimal transport to give it.

Wasserstein distances

Let $(\mathcal{X}, d)$ be a Polish metric space, and let $p \in [1, \infty)$. For two probability measures μ, ν on $\mathcal{X}$, the **Wasserstein distance of order p** between μ and ν is defined as follows:

$$W_p(\mu, \nu) = \inf\Big\{ \big[\mathbb{E} d(f, g)^p\big]^{\frac{1}{p}} \mid dist(f) = \mu, dist(g) = \nu \Big\},$$

where f, g are two $\mathcal{X}$-valued random variables.

Wasserstein spaces

Let $(\mathcal{X}, d)$ be a Polish metric space, and let $p \in [1, \infty)$. Then the **Wasserstein space of order** p is defined as

$$P_p(\mathcal{X}) := \{\mu \in \mathfrak{D}(\mathcal{X}) \mid \int_{\mathcal{X}} d(x_0, x)^p \mu(dx) < +\infty \text{ for some } x_0 \in \mathcal{X}\},$$

where $\mathfrak{D}(\mathcal{X})$ is the space of all probability measures on $\mathcal{X}$.

Type spaces and Wasserstein spaces

Theorem (S.)

Suppose $M = L^1\big((\Omega, \mathcal{F}, m), [0,1]\big)$. *Let* $f = (f_1, \cdots, f_n)$ *be an n-tuple in a model M of* ARV. *Then* $f_*(m)$ *is the pushforward probability measure on* $[0,1]^n$. *Moreover, the mapping*

$$\eta_n \colon S_n(\mathrm{ARV}) \to \mathfrak{D}\big([0,1]^n\big)$$
$$tp(f) \mapsto f_*(m)$$

is an isometric isomorphism between $\big(S_n(\mathrm{ARV}), d^*\big)$ *and* $\Big(\mathfrak{D}\big([0,1]^n\big), W_1\Big)$, *where the metric on* $[0,1]^n$ *is defined as* $d^*_{[0,1]^n}(c,d) = \sum_{i=1}^n |c_i - d_i|$, *for* $c, d \in [0,1]^n$.

Proposition

$(S_n(\text{ARV}),$ *logic topology*$) = (S_n(\text{ARV}),$ *metric topology*$)$, *i.e., two topologies coincide.*

Proof.

By the fact that ARV is separably categorical and Ryll-Nardzewski Theorem. □

Remark

Here, the logic topology corresponds to the weak convergence topology on $\mathfrak{D}([0,1]^n)$ *and the metric topology corresponds to the topology by the Wasserstein distance on* $\mathfrak{D}([0,1]^n)$. *This is a special case, when* $\mathcal{X} = [0,1]^n$, *of a proposition in Villani's book "Optimal Transport: Old and New".*

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Given that $\mathcal{M} \models \text{ARV}$ and $f = (f_1, \cdots, f_n) \in M^n$. Let $\text{ARV}(f)$ denote $Th(\mathcal{M}, f)$.

Proposition (S.)

$(S_n(\text{ARV}(f)),$ logic topology$) = (S_n(\text{ARV}(f)),$ metric topology$)$ if and only if f is discrete.

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This is related to d-finite tuples in ARV; see my paper: "On d-finite tuples in random variable structures", Fund. Math. 221 (2013), 221–230.

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Thanks!!

Thanks for your attention!