Introduction to continuous logic

Ultraproducts and type spaces

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Let I be a nonempty set. A *filter* on I is a collection F of subsets of I satisfying:

1. $\emptyset \notin F$ and $I \in F$.
2. $A, B \in F$ implies $A \cap B \in F$
3. $A \subseteq B \subseteq I$ and $A \in F$ implies $B \in F$.

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Example

- Let $I = \mathbb{N}$ and $F = \{A \mid |\mathbb{N} \backslash A| < \infty\}$. Then F is a *Frechét filter*. But, it's not an ultrafilter.
- Let $a \in I$ and $F = \{A \mid a \in A\}$. Then F is an ultrafilter on I, called the *principal ultrafilter on I generated by a*.
- Principal ultrafilters are trivial.

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Proof By Zorn's lemma.

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Definition of ultraproducts

- Fix a first order signature L. Let I be a nonempty set and let $\mathcal{U}$ be a fixed ultrafilter on I.
- Consider an I-indexed family of L-structure $\{\mathcal{A}_i \mid i \in I\}$.
- Let $A = \prod_{i \in I} A_i$ be the cartisian product of A_i.
- Let $f, g \in A$. Define a relation on A, $f \sim g$ iff $\{i \in I \mid f(i) = g(i)\} \in \mathcal{U}$.

Fact The relation $\sim$ is an equivalence relation on A.

- $A/\sim$ is the *ultraproduct of the sets A_i with respect to $\mathcal{U}$*, denoted by $\prod_{\mathcal{U}} A_i$.
- $\prod_{\mathcal{U}} A_i = \{[f]_{\mathcal{U}} = f/\sim \mid f \in \prod_{i \in I} A_i\}$.

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Ultraproducts of first order structures

The ultraproduct $\prod_{\mathcal{U}} \mathcal{A}_i$ is defined to be an L-structure.

- The universe is $\prod_{\mathcal{U}} A_i$.
- for each constant c in L, define f by $f(i) = c^{\mathcal{A}_i}$ for each $i \in I$. Then $c^{\prod_{\mathcal{U}} \mathcal{A}_i} = [f]_{\mathcal{U}}$.
- for each predicate P in L, $P^{\prod_{\mathcal{U}} \mathcal{A}_i}([f_1]_{\mathcal{U}}, \cdots, [f_n]_{\mathcal{U}})$ iff $\{i \in I \mid P^{\mathcal{A}_i}(f_1(i), \cdots, f_n(i))\} \in \mathcal{U}$.
- for each function F in L, $F^{\prod_{\mathcal{U}} \mathcal{A}_i}([f_1]_{\mathcal{U}}, \cdots, [f_m]_{\mathcal{U}}) = [f]_{\mathcal{U}}$, where $f \in A$ is defined by $f(i) = F^{\mathcal{A}_i}(f_1(i), \cdots, f_m(i))$.

An *ultrapower* of $\mathcal{A}$ is an ultraproduct $\prod_{\mathcal{U}} \mathcal{A}_i$ with $\mathcal{A}_i = \mathcal{A}$ for each $i \in I$, denoted by $\mathcal{A}^{\mathcal{U}}$.

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Fundamental theorem of ultraproducts

Theorem (Łos Theorem)

For every L-formula $\varphi(x_1, \cdots, x_n)$ *and every* $f_k \in \prod_{i\in I} A_i$ *for each* $1 \leq k \leq n$*, we have*

$$\prod_{\mathcal{U}} \mathcal{A}_i \models \varphi([f_1]_{\mathcal{U}}, \cdots, [f_n]_{\mathcal{U}})$$

iff

$$\{i \in I \mid \mathcal{A}_i \models \varphi(f_1(i), \cdots, f_n(i))\} \in \mathcal{U}.$$

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Remark

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- If $\mathcal{U}$ is an ultrafilter on I and $x \in X$, we write

$$\lim_{\mathcal{U}} x_i = x$$

and say that x is the *ultralimit* of $(x_i)_{i\in I}$ along $\mathcal{U}$ if for every open set F containing x, $\{i \in I \mid x_i \in F\} \in \mathcal{U}$.

Fact Let X be the compact Hausdorff space. Then the ultralimit always exists and is unique.

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Ultraproducts of metric spaces

- Let $\{(M_i, d_i) \mid i \in I\}$ be a family of bounded metric space with a fixed diameter K. Let $\mathcal{U}$ be an ultrafilter on I.
- Define d on $\prod_{i\in I} M_i$ by $d(x, y) = \lim_{\mathcal{U}} d_i(x_i, y_i)$, where $x = (x_i)_{i\in I}$ and $y = (y_i)_{i\in I}$.
- Check: d is a pseudometric on $\prod_{i\in I} M_i$.
- Let $x, y \in \prod_{i\in I} M_i$. Define a relation $x \sim y$ iff $d(x, y) = 0$.
- The relation $\sim$ is an equivalence relation.
- We define $(\prod_{i\in I} M_i)_{\mathcal{U}}$ as $(\prod_{i\in I} M_i)/\sim$. The space $((\prod_{i\in I} M_i)_{\mathcal{U}}, d)$ is a metric space, and indeed a complete metric space, called the *ultraproduct* of $\{(M_i, d_i) \mid i \in I\}$.

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Fact Every ultrapower of a closed bounded interval may be canonically identified with the interval itself.

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Ultraproducts of functions

Let $\mathcal{U}$ be an ultrafilter on I.

- Suppose $\{(M_i, d_i) \mid i \in I\}$ and $\{(M_i', d_i') \mid i \in I\}$ are families of metric spaces, with a fixed diameter K.
- Fix $n \geq 1$ and suppose $f_i \colon M_i^n \to M_i'$ is uniformly continuous for each $i \in I$.
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as follows:

$$(\prod_{i\in I} f_i)_{\mathcal{U}}([(x_i^1)_{i\in I}]_{\mathcal{U}}, \cdots, [(x_i^n)_{i\in I}]_{\mathcal{U}}) = [(f_i(x_i^1, \cdots, x_i^n))_{i\in I}]_{\mathcal{U}},$$

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Let $\{\mathcal{M}_i \mid i \in I\}$ be a family of L-structures, and let $\mathcal{U}$ be an ultrafilter on I. The ultraproduct $\prod_{\mathcal{U}} \mathcal{M}_i$ of L-structures is defined to be an L-structure $\mathcal{M}$ as follows:

- The universe is $M = (\prod_{i \in I} M_i)_{\mathcal{U}}$.
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Fundamental theorem of ultraproducts

Theorem (Łos Theorem for continuous logic)

Let $\varphi(x_1, \cdots, x_n)$ *be an L-formula. If* $a^k = [(a_i^k)_{i \in I}]_{\mathcal{U}}$ *are elements of M for each* $1 \leq k \leq n$*, then*

$$\varphi^{\mathcal{M}}(a^1, \cdots, a^n) = \lim_{\mathcal{U}} \varphi^{\mathcal{M}_i}(a_i^1, \cdots, a_i^n).$$

□

Theorem (Compactness Theorem)

Let T be an L-theory and let $\mathcal{C}$ be a class of L-structures. Assume that T is finitely satisfiable in $\mathcal{C}$. Then there is an ultraproduct of structures from $\mathcal{C}$ that is a model of T. □

Theorem (Downward Löwenheim-Skolem Theorem)

Let κ be an infinite cardinal and assume that $\mathrm{Card}(L) \leq \kappa$. Let $\mathcal{M}$ be an L-structure and suppose that $A \subseteq M$ has density character $\leq \kappa$. Then, there is an elementary substructure $\mathcal{N}$ of $\mathcal{M}$ such that

1. $\mathcal{N} \preceq \mathcal{M}$
2. $A \subseteq N \subseteq M$
3. *the density character of N is* $\leq \kappa$.

□

Theorem

Let T be a complete L-theory and let κ be an infinite cardinal. Then T has a κ-universal domain, i.e., a κ-saturated and strongly κ-homogeneous model. □

Counter Examples

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- For each $n \in \mathbb{N}^+$, define a discontinuous function $f_n \colon [0,1] \to [0,1]$ as follows:

$$f_n(x) = \begin{cases} 0 & 0 \le x < \frac{1}{n} \\ 1 & \frac{1}{n} \le x \le 1 \end{cases}$$

- Note that $(0,0,0,\cdots,0,\cdots) \sim (0,1,\frac{1}{2},\cdots,\frac{1}{n},\cdots)$, since $\lim_{\mathcal{U}} \frac{1}{n} = 0$.
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- It is necessary and sufficient that the function f is uniformly continuous.
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Let $\mathcal{U}$ be a nonprincipal ultrafilter on $\mathbb{N}^+$.

- For each $n \in \mathbb{N}^+$, define $f_n \colon [0,1] \to [0,1]$ as follows:

$$f_n(x) = \begin{cases} nx & 0 \leq x < \frac{1}{n} \\ 1 & \frac{1}{n} \leq x \leq 1 \end{cases}$$

- These f_n's are uniformly continuous functions, but have different modulus of uniform continuity Δ_{f_n}.
- Note that $(0,0,0,\cdots,0,\cdots) \sim (0,1,\frac{1}{2},\cdots,\frac{1}{n},\cdots)$, since $\lim_{\mathcal{U}} \frac{1}{n} = 0$.
- $\prod_{\mathcal{U}} f_n((0,0,0,\cdots,0,\cdots)) = 0$
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Definition

- Let $\mathcal{M} \models T$ and $A \subseteq M$. Denote $L(A)$-structure $(\mathcal{M}, a)_{a \in A}$ by $\mathcal{M}_A$ and denote the theory $\mathrm{Th}_{L(A)}(\mathcal{M}_A)$ by T_A.
- A set p of $L(A)$-conditions with n free variables $x = (x_1, \cdots, x_n)$, is called a (complete) *n-type over A* if there exists a model $\mathcal{M}_A \models T_A$ and $e \in M^n$ such that p is the set of all $L(A)$-conditions $E(x)$ for which $\mathcal{M}_A \models E(e)$.
- We denote p by $\mathrm{tp}(e/A)$ and say that e *realizes p in* $\mathcal{M}$.
- The collection of all such n-types over A is denoted by $S_n(T_A)$, or simply $S_n(A)$.

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The logic topology on types

- Let $\varphi(x_1, \cdots, x_n)$ be an $L(A)$-formula and let $\epsilon > 0$.
- $[\varphi < \epsilon] = \{p \in S_n(T_A) \mid$ for some $0 \leq \delta < \epsilon$, the condition $(\varphi \leq \delta) \in p\}$.
- The *logic topology* on $S_n(T_A)$ is defined as follows: If $p \in S_n(T_A)$, then the basic neighborhoods of p are the set of the form $[\varphi < \epsilon]$ for which the condition $\varphi = 0$ is in p and $\epsilon > 0$.

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Theorem

The logic topolgy on $S_n(T_A)$ is compact and Hausdorff. □

The metric topology on types

- Let $\mathcal{M}_A$ be a model of T_A such that each type in $S_n(T_A)$ is realized.
- For $p, q \in S_n(T_A)$, we define $d(p, q)$ to be

$$\inf\{\max_{1\leq j\leq n} d(b_j, c_j) \mid \mathcal{M}_A \models p[b_1, \cdots, b_n], \mathcal{M}_A \models q[c_1, \cdots, c_n]\}.$$

Claim The definition of $d(p, q)$ does not depend on $\mathcal{M}_A$.

- Note that $(S_n(T_A), d)$ is a metric space, and d induces a metric topology on $S_n(T_A)$.

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On $S_n(T_A)$, the metric topology is finer than the logic topology. □

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Ryll-Nardzewski Theorem-continuous versoin

Theorem (Henson)

Let T be a complete countable theory. Then the following are equivaent:

1. *T is separably categorical, i.e., T has a unique separable model up to isomorphism.*
2. *For each n, the metric space $(S_n(T), d)$ is compact.*
3. *For each n, the logic topology and the metric topology coincide on $S_n(T)$.*

Separably categorical theories

The following theories are separably categorical:

- The theory of infinite dimensional Hilbert spaces.
- The theory of the Urysohn sphere.
- The theory of atomless probabilty algebras.

Functions on type spaces

- Let $\mathcal{M}_A$ be a model of T_A in which every type in $S_n(T_A)$ is realized for each $n \geq 1$.
- Let $\varphi(x_1, \cdots, x_n)$ be an $L(A)$-formula.
- For each $p \in S_n(T_A)$, define $\tilde{\varphi} = r$, where r is the unique real number such that $\varphi = r$ is in p.
- Equivalently, $\tilde{\varphi}(p) = \varphi^{\mathcal{M}_A}(b)$ where $b \models p$.

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Theorem

Let $\varphi(x)$ be an $L(A)$-formula. Then $\tilde{\varphi}\colon S_n(T_A) \to [0,1]$ is continuous with respect to the logic topology and uniformly continuous with respect to the metric topology.

Thanks!!

Thanks for your attention!