Introduction to continuous logic

Syntax, and semantics

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- Continuous logic, a.k.a., continuous first order logic, continuous model theory, model theory for metric structures.
- The truth values are not just $\{T, F\}$, but $[0, 1]$.
  The quantifiers are inf and sup.
- Chang and Keisler's continuous model theory in the 1960s and Łukasiewicz logic were the early attempts to deal with non-classical logic.
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Henson's logic; positive bounded formulas with an approximate semantics.

1981 Krivine and Maurey, Espaces de Banach stables, Israel J. Math. 39 (1981), 273–295.
Every infinite dimensional stable Banach space contains l^p, for some p, $1 \leq p < \infty$.

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Some history

1930s	Compactness Theorem, Ultraproducts, Saturation
1960s	Ax-Kochen, Ershov, Diophantine problems over local fields
	Abraham Robinson, Nonstandard analysis
1970s	Shelah, Classification Theory, Stability Theory
1996	Hrushovski, Mordell-Lang conjecture
1980s	O-minimal Theory
2011	Pila, André-Oort conjecture
2000s	Continuous Logic
2020s	Major breakthrough?

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Metric structures

- Let (M, d) be a complete bounded metric space.
- A *predicate* on M is a uniformly continuous function from M^n to $I_P = [a, b] \subseteq \mathbb{R}$, for some $n \geq 1$.
- $P\colon M^n \to \mathbb{R}$ is uniformly continuous if

$$\forall \epsilon > 0 \exists \delta > 0 \forall x, y \in M^n (d(x, y) < \delta \to |P(x) - P(y)| < \epsilon).$$

- A *function* on M is a uniformly continuous function from M^n to M for some $n \geq 1$.

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- For simplicity, (M, d) is bounded by 1, and predicates have values on $[0, 1]$, the truth values.
- In first order logic, predicates $M^n \to \{0, 1\}$.
- In continuous logic, predicates $M^n \to [0, 1]$.

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Definition

A *metric structure* $\mathcal{M}$ based on complete bounded metric space (M, d) consists of a family of $(P_i \mid i \in I)$ of predicates on M, a family of $(F_j \mid j \in J)$ of functions on M, and a family of $(a_k \mid k \in K)$ of distinguished elements of M.

We denote a metric structure as

$$\mathcal{M} = (M, P_i, F_j, a_k \mid i \in I, j \in J, k \in K).$$

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Examples

1. A complete bounded metric space (M, d) with no additional structures.
2. Given a first order structure $\mathcal{M}$. Define a discrete metric on M by $d(a, b) = 1$ if $a \neq b$, and $d(a, b) = 0$ if $a = b$. Then, $\mathcal{M}$ becomes a metric structure.

   This example shows that continuous logic is a generalization of first order logic.
3. Probability algebras are boolean algebras of events in probability space. We will discuss it further later.

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Classical Logic and Continuous Logic

	classical logic	continuous logic
truth values	$\{T, F\} = \{0, 1\}$	$[0, 1]$
quantifiers	$\forall x, \exists x$	$\sup x, \inf x$
functions	$M^n \to M$	$M^n \to M$
predicates	$M^n \to \{0, 1\}$	$M^n \to [0, 1]$
connectives	$\{0, 1\}^n \to \{0, 1\}$	$[0, 1]^n \to [0, 1]$
equality	$x = y$	$d(x, y) = 0$

Signature

- A signature or language L for continuous logic consists of symbols for constants, functions, and predicates, as usual.
  - constant symbols: interpreted as distinguished elements of M.
  - m-ary function symbols: interpreted as functions $f\colon M^m \to M$.
  - n-ary predicate symbols: interpreted as functions $P\colon M^n \to I_P$.
- L specifies a modulus of uniform continuity for each function symbol and each predicate symbol; more details on next slide.
- The metric is considered as a binary predicate (exactly as equality is used in classical logic).
- L provides D_L, the diameter of (M, d), and bounded interval I_P for each predicate P.
  For simplicity, assume $D_L = 1$ and $I_P = [0, 1]$.

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Modulus of uniform continuity

- For a function symbol f, the modulus of uniform continuity is a function $\Delta_f : (0,1] \to (0,1]$ satisfying $\forall \epsilon > 0$ $\forall x, y \in M^n$, if $d(x,y) < \Delta_f(\epsilon)$ then $d(f(x), f(y)) < \epsilon$.
- For a predicate symbol P, the modulus of uniform continuity Δ_P is defined similarly.
- The modulus of uniform continuity can be arranged to be an increasing continuous function $\Delta : (0,1] \to (0,1]$ so that $\lim_{t \to 0} \Delta(t) = 0$.
- Why uniform continuity is so important?

  Ultraproduct constructions.

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Terms and Formulas

- Terms: Terms are formed inductively, exactly as in first-order logic. Each variable and constant symbol is an L-term. If f is an n-ary function symbol and $t_1, \cdots, t_n$ are L-terms, then $f(t_1, \cdots, t_n)$ is an L-term. All L-terms are constructed in this way.
- Atomic formulas: The expressions of the form $P(t_1, \cdots, t_n)$, in which P is an n-ary predicate symbol of L and $t_1, \cdots, t_n$ are L-terms; as well as $d(t_1, t_2)$, in which t_1 and t_2 are L-terms.

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Formulas

The class of L-formulas is the smallest class of expressions satisfying the following requirements:

- Atomic formulas of L are L-formulas.
- If $u : [0, 1]^n \to [0, 1]$ is continuous and $\varphi_1, \cdots, \varphi_n$ are L-formulas, then $u(\varphi_1, \cdots, \varphi_n)$ is an L-formula.
- If φ is an L-formula and x is a variable, then $\sup_x \varphi$ and $\inf_x \varphi$ are L-formulas.

The closed formulas are called sentences.

Remark

So far, the definition of formulas is not a good one.

- Too general.

  There are uncountably many continuous functions; a dense subset will be enough.

- Too restrict.

  Need formulas closed under taking certain limits, in order to develop a good notion of "definibility".

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A motivation example

Example

Let D_0 be the set of repeating decimals. Then (D_0, d) is a *pseudometric space*.

- $d(0.\dot{9}, 1) = 0$, but $0.\dot{9} \neq 1$.
- Consider $(D, d) = (D_0, d)/\sim$, where $x \sim y$ if $d(x, y) = 0$.
- Then (D, d) is a metric space, but it is not complete.
- Take its completion to get $(\bar{D}, d)$.
- Note that $(D, d) = (\mathbb{Q}, d)$, and $(\bar{D}, d) = (\mathbb{R}, d)$.
- This example shows that how to start with a pseudometric space, to get a metric space, and a complete metric space.

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- Consider $(D, d) = (D_0, d)/\sim$, where $x \sim y$ if $d(x, y) = 0$.
- Then (D, d) is a metric space, but it is not complete.
- Take its completion to get $(\bar{D}, d)$.
- Note that $(D, d) = (\mathbb{Q}, d)$, and $(\bar{D}, d) = (\mathbb{R}, d)$.
- This example shows that how to start with a pseudometric space, to get a metric space, and a complete metric space.

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Prestructures

Fix a signature L. Let (M_0, d) be a pseudometric space, satisfying $\mathrm{diam}(M_0, d) \leq D_L$.

An *L-prestructure* $\mathcal{M}_0$ based on (M_0, d) is a structure satisfying:

1. for each predicate symbol P of L, $P^{\mathcal{M}_0} : M_0^n \to I_P$ has Δ_P as a modulus of uniform continuity.
2. for each function symbol f of L, $f^{\mathcal{M}_0} : M_0^m \to M_0$ has Δ_f as a modulus of uniform continuity.
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Quotients

Given an L-prestructure $\mathcal{M}_0$, we define its *quotient* as follows:

1. Let $(M, d) = (M_0, d)/\sim$, where $x \sim y$ iff $d(x, y) = 0$.
2. Let $\pi \colon M_0 \to M$ be the quotient map. Then
   (i) for each predicate symbol P, define $P^{\mathcal{M}} \colon M^n \to I_P$ by $P^{\mathcal{M}}(\pi(x_1), \cdots, \pi(x_n)) = P^{\mathcal{M}_0}(x_1, \cdots, x_n)$ for all $x \in M_0^n$.
   (ii) for each function symbol f, define $f^{\mathcal{M}} \colon M^m \to M$ by $f^{\mathcal{M}}(\pi(x_1), \cdots, \pi(x_m)) = \pi\left(f^{\mathcal{M}_0}(x_1, \cdots, x_m)\right)$ for all $x \in M_0^m$.
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Completion

Finally, we take the completion of $\mathcal{M}$ to get an *L-structure* $\mathcal{N}$.

1. for each P, define $P^{\mathcal{N}} \colon N^n \to I_P$ as the unique extension of $P^{\mathcal{M}}$ with the same Δ_P.
2. for each f, define $f^{\mathcal{N}} \colon N^m \to N$ as the unique extension of $f^{\mathcal{M}}$ with the same Δ_f.
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Then (N, d) is a bounded complete metric space and call $\mathcal{N}$ an *L-structure*.

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Semantics

- Let $\mathcal{M}$ be an L-prestructure, and let $A \subseteq M$. We extend L to a signature $L(A)$ by adding new constant symbols $c(a)$ for all $a \in A$.
- Interpret $c(a)^{\mathcal{M}} = a$ for each $a \in A$.
- Consider an $L(M)$-term $t(x_1, \cdots, x_n)$. Define $t^{\mathcal{M}} : M^n \to M$ exactly as in first order logic to interpret t in $\mathcal{M}$.

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Key definition of semantics in continuous logic

1. $(d(t_1, t_2))^{\mathcal{M}} = d^{\mathcal{M}}(t_1^{\mathcal{M}}, t_2^{\mathcal{M}})$ for all terms t_1, t_2.
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3. for all $L(M)$-sentences $\sigma_1, \cdots, \sigma_n$ and all continuous functions $u\colon [0,1]^n \to [0,1]$,

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Let $\varphi(x)$ be an $L(M)$-formula. Let $\varphi^{\mathcal{M}}$ denote the function $M^n \to [0,1]$ defined by

$$\varphi^{\mathcal{M}}(a) = (\varphi(a))^{\mathcal{M}}$$

for all $a \in M^n$.

Fact

$\varphi^{\mathcal{M}}$ is a uniformly continuous function.

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- Two L-formulas $\varphi(x)$ and $\psi(x)$ are *logical equivalent* if

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Definable predicates

- A mapping $P\colon M^n \to [0,1]$ is *a definable predicate* in $\mathcal{M}$ over A, if there is a sequence $(\varphi_k(x) \mid k \in \mathbb{N})$ of $L(A)$-formulas such that $\varphi_k^{\mathcal{M}}(x) \rightrightarrows P(x)$ on M^n.
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Size of the space of L-formulas

- The space of L-formulas is too big, since there are uncountably many connectives.
- We could consider the *density character* of the space, which is the smallest dense subset with respect to the logical distance between L-formulas.
- By Stone-Weierstrass Theorem, there is a countable set of functions $[0,1]^n \to [0,1]$ that is dense in the set of all continuous functions with respect to sup-distance. We may use this countable set of functions to build formulas, called restricted formulas.
- The size of restricted formulas is $\leq \mathrm{Card}(L)$.

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- By Stone-Weierstrass Theorem, there is a countable set of functions $[0,1]^n \to [0,1]$ that is dense in the set of all continuous functions with respect to sup-distance. We may use this countable set of functions to build formulas, called restricted formulas.
- The size of restricted formulas is $\leq \mathrm{Card}(L)$.

  Every *L*-formula can be approximated arbitrarily closely in logical distance by a restricted formula.

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Connectives

The set $C([0,1]^n,[0,1])$ is uncountable, we would rather consider a countable dense subset of $C([0,1]^n,[0,1])$. The following 3 connectives can generate a dense family of connectives.

- $\neg x = 1 - x$
- $x \dot{-} y = \max\{x - y, 0\}$
- $\frac{1}{2}x = x/2$

e.g.

- $x \wedge y = \min\{x, y\} = x \dot{-} (x \dot{-} y)$
- $x \vee y = \max\{x, y\} = \neg x(\neg x \wedge \neg y)$
- $|x - y| = (x \dot{-} y) \vee (y \dot{-} x)$

Conditions

- Conditions: An L-condition E is a formal expression of the form $\varphi = 0$, where φ is an L-formula.
- Closed conditions: We call a condition E is closed if φ is a sentence.

Theory

Definition

- A theory in L is a set of closed L-conditions. If T is a theory in L and M is an L-structure, we say that M is a model of T and write $M \models T$ if $M \models E$ for every condition E in T.
- If M is an L-structure, the theory of M, denoted by $Th(M)$, is the set of closed L-conditions that are true in M. If T is a theory of this form, it will be called complete.

A continuous model theory has compactness theorem, Löwenheim-Skolem theorem and existence of saturated and homogeneous models as classic model theory.

Thanks!!

Thanks for your attention!