Graphs on Infinite Cardinals: The Erdős–Dushnik–Miller theorem

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Ordinal and Cardinal

What is the smallest number that is 'greater than' every natural number?

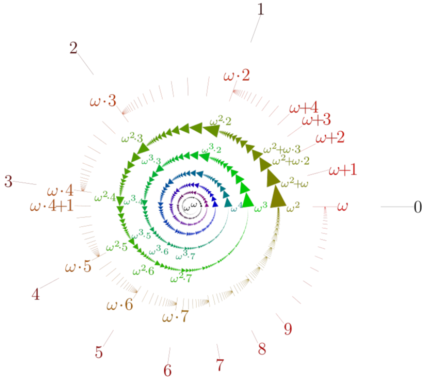
Definition

Define $0 := \varnothing$, $1 := \{0\}$, and $n := \{0, 1, \ldots, n-1\}$. *The set of natural numbers* $\mathbb{N}$ is the smallest set containing 0 and closed under successor.

The usual ordering $n < m$ is equivalent to $n \in m$.

Definition (Informal)

ω is the smallest number which is greater than every natural number, which is defined as $\{0, 1, \ldots\}(= \mathbb{N})$. Ordinals are defined as the same rule, e.g., $\alpha^+ = \alpha \cup \{\alpha\}$.



Facts

1. *(AC) Every set can be enumerated using an ordinal index.*
2. $\in$ *in ordinals is well-ordering, i.e., any subset has a least element.*
3. *Thus, there is no infinite descending chain of ordinals.*

Some ordinal numbers are special, in the sense that they can measure the 'size' of a set.

Definition

Card(A) denotes the least ordinal α such that there is a bijective function between α and A.

For example, ω is a cardinal that measures countable set.
But ordinals $\omega + 1$ and ω^ω are not a cardinal, since $\omega + 1 \approx \omega \approx \omega^\omega$.

From now on, κ denotes an infinite cardinal.

Motivation

Exercise (I.19, Set Theory by Kenneth Kunen)

Let κ be an infinite cardinal and $\lhd$ any well-ordering of κ. Show that there is an $X \subset \kappa$ such that $|X| = \kappa$, and $\lhd$ and $\in$ agree on X.

e.g. $\kappa = \aleph_1$, the first uncountable cardinal.

$\longrightarrow$ increasing

$\in$-order	0	1	2	$\cdots$	ω	$\omega+1$	$\cdots$	$\omega \cdot 2$	$\cdots$	ω^ω	$\cdots$
$\lhd$-order	ω^ω	3	ω	0	ω^2+4	$\omega+1$	ω^7	ω^{ω^ω}	$\cdots$		

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$\in$-order: $0 \quad 1 \quad 2 \quad \cdots \quad \omega \quad \omega{+}1 \quad \cdots \quad \omega \cdot 2 \quad \cdots \quad \omega^\omega \quad \cdots$

$\lhd$-order: $\omega^\omega \quad \boxed{3} \quad \boxed{\omega} \quad 0 \quad \omega^2 + 4 \quad \omega{+}1 \quad \boxed{\omega^7} \quad \boxed{\omega^{\omega^\omega}} \quad \cdots$

Find an increasing $\lhd$-chain of size κ while no infinite decreasing $\lhd$-chain.

Ramsey Property?

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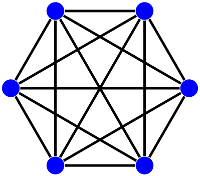
Let κ be an infinite cardinal and $\lhd$ any well-ordering of κ. Show that there is an $X \subset \kappa$ such that $|X| = \kappa$, and $\lhd$ and $\in$ agree on X.

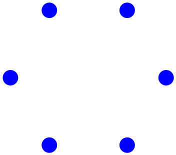
Define a graph G on κ, where α and β are adjacent if their orders agree.

$$E(\alpha, \beta) = (\alpha \in \beta \wedge \alpha \lhd \beta) \ \vee \ (\beta \in \alpha \wedge \beta \lhd \alpha)$$

Increasing $\lhd$-chain	$\Leftrightarrow$	Clique
Decreasing $\lhd$-chain	$\Leftrightarrow$	Independent set



(a) A clique



(b) An independent set

Exercise (I.19, Set Theory by Kenneth Kunen)

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Theorem (Erdős–Dushnik–Miller, 1941)

Any graph G on an infinite cardinal κ has a size κ clique or a size ω independent set. (Equivalently, κ independent set or ω clique)

Or, using arrow notation,

Theorem (Erdős–Dushnik–Miller, 1941)

$\kappa \to (\kappa, \omega)^2$ holds, i.e., a 2-coloring of κ^2 has a monochromatic subset of size κ or ω.

It is an unbalanced generalizaion of the Ramsey theorem, $\omega \to (\omega)^r_k$.

Proof of EDM Theorem

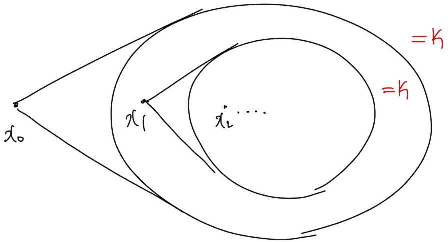
Claim

Let $G = (V, E)$ be a graph on $V = \kappa$. If G does not have a size κ independent set, then there is a clique of size ω.

Lemma

If G does not have a size κ independent set, then there exists a vertex adjacent to κ many elements.

Proof of the claim.



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Assume there is no such vertex. Define $N_H(x)$ a neighborhood of x in H.

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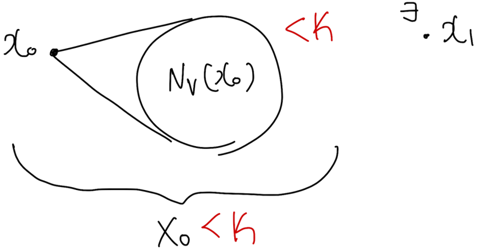

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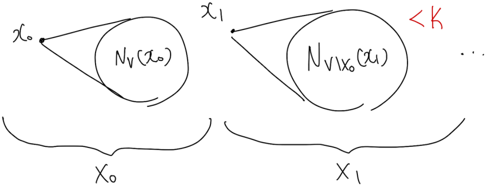

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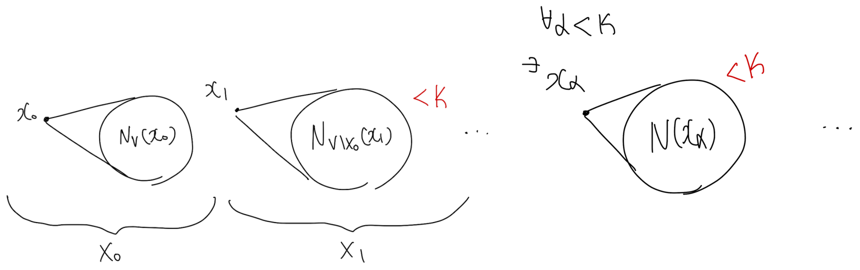

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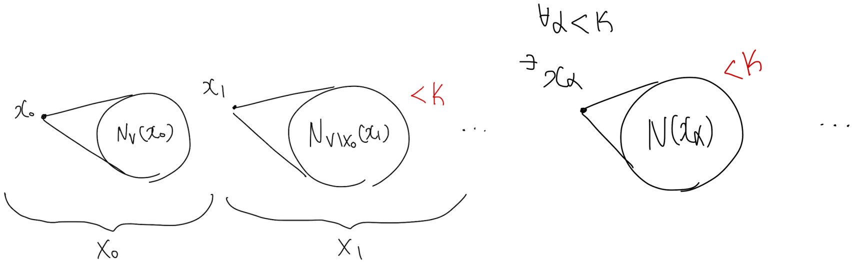
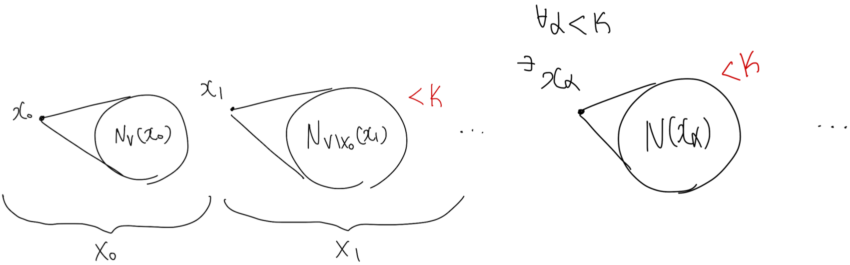


$\{x_\alpha : \alpha \in \kappa\}$ is an independent set. Contradiction. □

Wait, are we sure that we can choose x_α for all $\alpha \in \kappa$?



What if vertices run out in the middle?

$$\exists\ \alpha \in \kappa \text{ such that } \operatorname{Card}\left(\bigcup_{\gamma=0}^{\alpha} X_\gamma\right) = \kappa.$$

Such cardinals are called *singular*, e.g, $\aleph_\omega = \bigcup_{n \in \omega} \aleph_n$.

If it does not happen, it is called *regular*.

Proof of EDM theorem: regular case

Claim

Let $G = (V, E)$ be a graph on $V = \kappa$, a regular cardinal. If G does not have a size κ independent set, then there is a clique of size ω.

Lemma

Let κ be a regular cardinal. If G does not have a size κ independent set, then there exists a vertex adjacent to κ many elements.

For a singular case, it is still true, but we need more (complicated) argument.

Theorem (Erdős–Dushnik–Miller, 1941)

$\kappa \to (\kappa, \omega)^2$ *holds.*

Theorem (Erdős–Dushnik–Miller, 1981)

$\kappa \to (\kappa, \omega + 1)^2$ *holds for an uncountable regular cardinal* κ.

Proof of EDM theorem '81

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Proof stekch.

Let $G = (V, E)$ be a graph on $V = \kappa$. Assume no size κ independent set. Define a tree $\langle T, \prec \rangle$ of length ω, and a function $S(X) \subseteq \kappa$ for each $X \in T$, where X is a maximal independent subset of $S(X)$.

Proof of EDM theorem '81

Theorem (Erdős–Dushnik–Miller, 1981)

$\kappa \to (\kappa, \omega + 1)^2$ *holds for an uncountable regular cardinal* κ.

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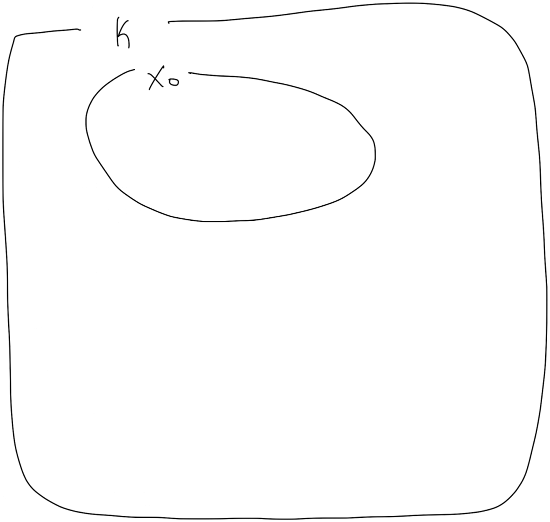
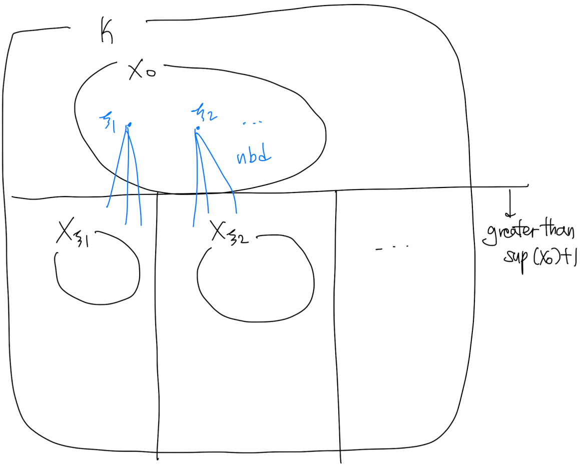
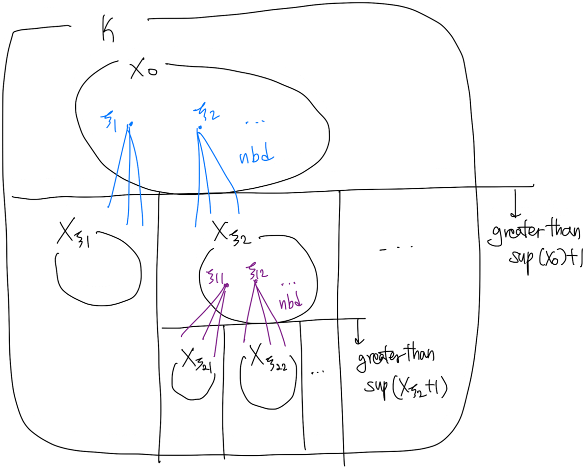
Let $G = (V, E)$ be a graph on $V = \kappa$. Assume no size κ independent set. Define a tree $\langle T, \prec \rangle$ of length ω, and a function $S(X) \subseteq \kappa$ for each $X \in T$, where X is a maximal independent subset of $S(X)$. Namely,

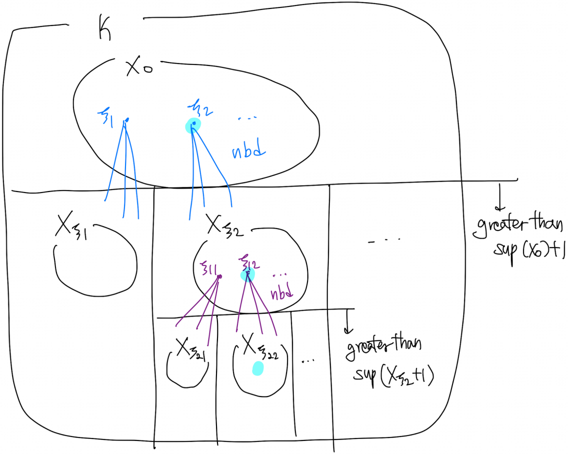
1. X_0 is a maximal independent set of V, and $S(X_0) = V$.
2. For each $\xi \in X \in T[n]$, put $S_\xi(X) = N_V(\xi) \cap (S(X) \setminus (\sup(X) + 1))$.
3. Let X_ξ be a maximal independent set of $S'_\xi(X) = S_\xi(X) \setminus \bigcup_{\eta < \xi, \eta \in X} S_\eta(X)$.
4. Put $\{X_\xi : \xi \in X\}$ at $T[n+1]$, and let $S(X_\xi) = S'_\xi$.


K
X₀

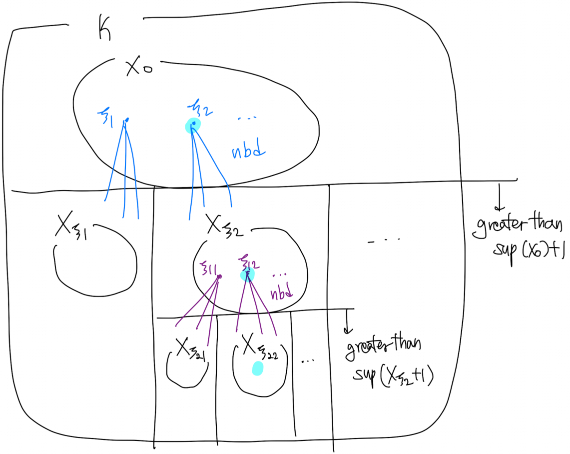

κ
X_0
ξ_1
ξ_2
...
nbd
X_{ξ_1}
X_{ξ_2}
...
greater than
sup $(X_0)+1$



K
X₀
ξ1
ξ2
...
nbd
Xξ1
Xξ2
ξ11
ξ12
...
nbd
greater than
sup(X₀)+1
greater than
sup(Xξ2+1)



κ
X_0
ξ_1
ξ_2
nbd
X_{ξ_1}
X_{ξ_2}
ξ_{11}
ξ_{12}
nbd
$X_{\xi_{21}}$
$X_{\xi_{22}}$
greater than sup $(X_0)+1$
greater than sup $(X_{\xi_2}+1)$



We can prove T has an infinite branch $H = \{X^n\}_{n<\omega}$ such that

$$\bigcap\{S(X) : X \in H\} \neq 0.$$

Then $\{\xi_n\}_{n\in\omega+1}$ is a clique of order type $\omega + 1$.

Further Results

Question

Which singular cardinal κ satisfies $\kappa \to (\kappa, \omega+1)^2$?

1. If $\mathrm{cf}(\kappa) = \omega$, then it does not hold.
2. If $\mathrm{cf}(\kappa) > \omega$ and κ is a strong limit cardinal, i.e., $\forall \lambda < \kappa (2^\lambda < \kappa)$, then it is true.
3. In 2009, Shelah proved that if $\mathrm{cf}(\kappa) > \omega$ and $2^{\mathrm{cf}(\kappa)} < \kappa$, then it is true.

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