The 4th Korea Logic Day 2025

Formalizing intuitionistic negations in natural deduction systems

Seungrak Choi
Philosophy Major
The College of Humanities at Hallym University

HALLYM UNIVERSITY

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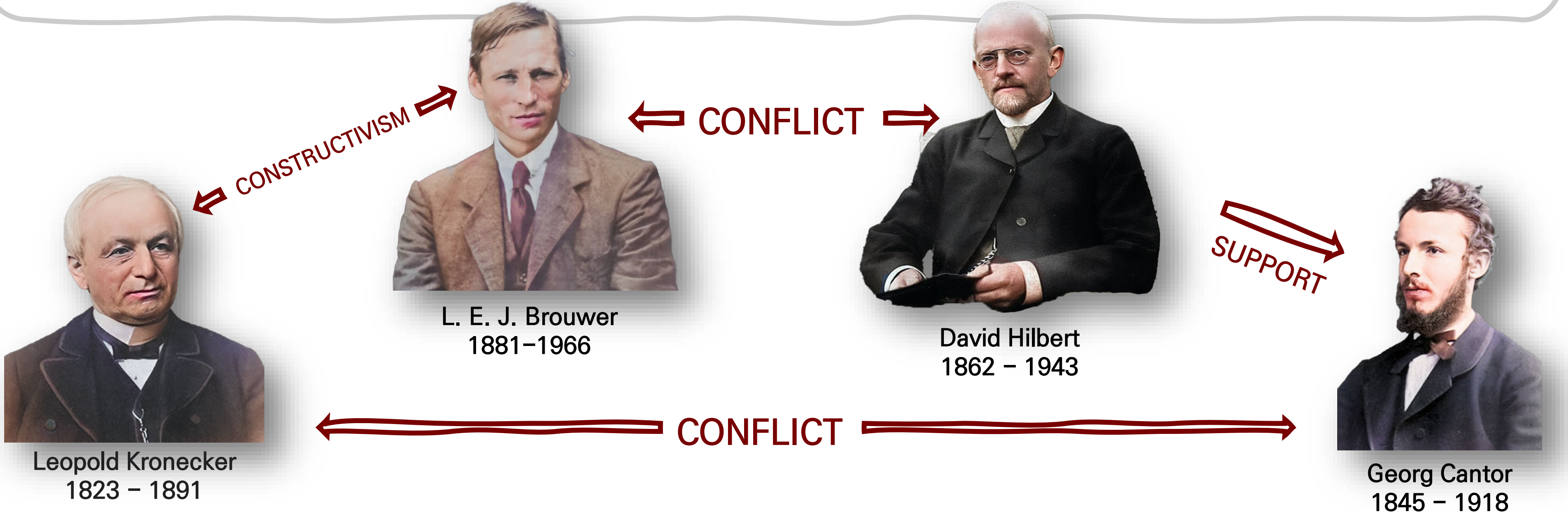
2.1.1. Luitzen Egbertus Jan Brouwer and His Intuitionism

L. E. J. Brouwer
1881. 2.27 – 1966.12.2

Historical Context & Background

1881: Born in Overschie, Netherlands.
1907: Earned Ph.D. in Mathematics with a dissertation on the foundations of mathematics, initiating his intuitionistic approach to mathematics.
1909–1913: Founded modern topology, introducing groundbreaking theorems such as the invariance of dimension and the fixed-point theorem.
1912: Appointed professor at the University of Amsterdam; his inaugural lecture, Intuitionism and Formalism, established the foundation for intuitionistic mathematics.
1920s: Engaged in foundational debates with David Hilbert, challenging the formalist approach, and published intuitionistic set theory, reshaping mathematical philosophy.



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L. E. J. Brouwer
1881. 2.27 – 1966.12.2

Brouwer's Philosophy of Mathematics: Mathematics as Mental Construction

1. Mathematics is a free creation of the human mind, where mathematical truths do not exist independently in the external world but are actively created through our mental activities.
2. Mathematics is independent of language and Platonic realms, as it exists in the mind prior to any linguistic expression and can be understood without the need for language.
3. Based on pure intuition of time (Kantian influence), mathematics fundamentally relies on our recognition of time's flow, where the experience of one moment leading to another serves as the foundation for mathematical thinking.
4. "There are no non-experienced truths."(Brouwer, 1975), meaning that all mathematical truths must be mentally experienced and cannot be established merely showing it cannot be false

"FIRST ACT OF INTUITIONISM. Completely separating mathematics from mathematical language and hence from the phenomena of language described by theoretical logic, recognizing that intuitionistic mathematics is an essentially languageless activity of the mind having its origin in the perception of a move of time. This perception of a move of time may be described as the falling apart of a life moment into two distinct things, one of which gives way to the other, but is retained by memory. If the twoity thus born is divested of all quality, it passes into the empty form of the common substratum of all twoities. And it is this common substratum, this empty form, which is the basic intuition of mathematics."

Brouwer, L.E.J. *Brouwer's Cambridge Lectures on Intuitionism*. Edited by D. van Dalen. Cambridge: Cambridge University Press, 1981, p.4.

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4. “There are no non-experienced truths.”, meaning that all mathematical truths must be mentally experienced and cannot be established merely showing it cannot be false.

The ... point of view that there are no non-experienced truths and that logic is not an absolutely reliable instrument to discover truths, has found acceptance with regard to mathematics much later than with regard to practical life and to science. Mathematics rigorously treated from this point of view, and deducing theorems exclusively by means of introspective construction, is called intuitionistic mathematics. In many respects it deviates from classical mathematics.

L.E.J. Brouwr (1948), “Consciousness, philosophy, and mathematics” in Heyting, A. (ed.), *Collected Works: Philosophy and Foundations of Mathematics*, Vol. 1. Elsevier, 1975, p. 488.

2. Some Histories

2.1. History of 'Constructive Proof': L.E.J. Brouwer

2.1.2. Brouwer's Notion of a Constructive Proof

L.E.J. Brouwer Giving a Lecture
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Brouwer did not explicitly introduce the modern term "constructive proof," his recurrent use of expressions such as "introspective construction" and the imperative that all mathematical objects be "built up" in consciousness illuminated much of what we now identify as constructive reasoning.

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Here, "no non-experienced truths" can be understood as the stance that no truth exists unless one can provide a recognizable or demonstrable way to establish it—a perspective that places the onus on constructive verification rather than invoking the law of the excluded middle, which classical mathematics often assumes.

The Fundamental Principle of Brouwer's Intuitionism:

For something to exist mathematically, one must explicitly construct it or present an effective method for its construction, rather than merely showing it cannot be false, as is done with the law of the excluded middle.

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The Law of Exlucded Middle (LEM): For any A, $A \lor \neg A$

$$\dfrac{\begin{matrix}[A] & \qquad & [\neg A]\\ \vdots & & \vdots \\ C & & C\end{matrix}}{C}\,\text{Dil}$$

LEM is often understood as treating $A \lor \neg A$ as a logical truth while applying the rule of or-elimination.

$$\dfrac{A \lor \neg A \qquad \begin{matrix}[A]^1\\ \vdots\\ C\end{matrix} \qquad \begin{matrix}[\neg A]^2\\ \vdots\\ C\end{matrix}}{C}\,\lor E,\ 1,\ 2$$

Dilemma rule is often regarded as LEM, but it’s different.

Premise 1: If studying logic is essential for AI research, we will study logic.
Premise 2: If studying logic is not essential for AI research, we will still study logic.
(Hidden?) Premise: Either studying logic is essential for AI research or it is not.
Conclusion: Therefore, we will study logic.

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Classical Reductio ad Absurdum (CR)

$$\dfrac{\begin{array}{c}[\neg A]^1 \\ \vdots \\ \bot\end{array}}{A}\,\text{CR},1 \qquad \dfrac{\neg\neg A}{A}\,\text{DNE}$$

It is known as the rule of *Classical Reductio ad Absurdum* because it allows the derivation of A from the double negation ¬¬A. This rule is also commonly referred to as the *Double Negation Elimination Rule*.

Neil Tennant often regarded the rule of "proof by contradiction" as a form of "intuitionistic reductio ad absurdum."

Neil Tennant Giving a Lecture
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Reductio ad Absurdum (Reduction to Absurdity, Proof by Contradiction)

$$\dfrac{\begin{array}{c}[A]^1 \\ \vdots \\ \bot\end{array}}{\neg A}\,\neg\text{I},1$$

"reductio" – "reduction" or "bringing back."
"ad" – "to" or "toward."
"absurdum" – "absurdity" or "nonsense."

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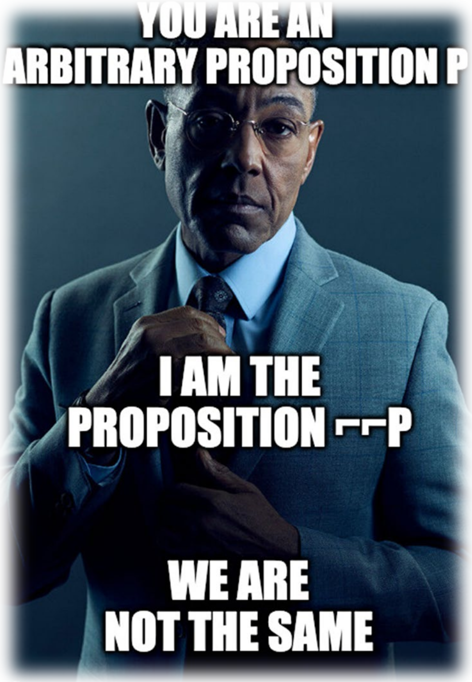
LEM, CR, and DNE exemplify classical inference rules, as they permit the derivation of A from its double negation ¬¬A. However, Brouwer argued that this inference is not universally valid. Fundamentally, he rejected the notion that ¬¬A and A are always equivalent.

$$\dfrac{\begin{matrix}[\neg A]^1\\ \vdots\\ \bot\end{matrix}}{A}\ \text{CR},1 \qquad \dfrac{\begin{matrix}[A]\\ \vdots\\ C\end{matrix}\qquad\begin{matrix}[\neg A]\\ \vdots\\ C\end{matrix}}{C}\ \text{Dil} \qquad \dfrac{\neg\neg A}{A}\ \text{DNE} \qquad \dfrac{}{A\vee\neg A}\ \text{LEM}$$



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The Example of a Non-Constructive Proof.

Proposition. There exists two real numbers x and y such that x^y is rational.

Proof. Since $\sqrt{2}$ is an irrational (real) number, the expression $\sqrt{2}^{\sqrt{2}}$ must be either rational or irrational.

Case 1: $\sqrt{2}^{\sqrt{2}}$ is rational.

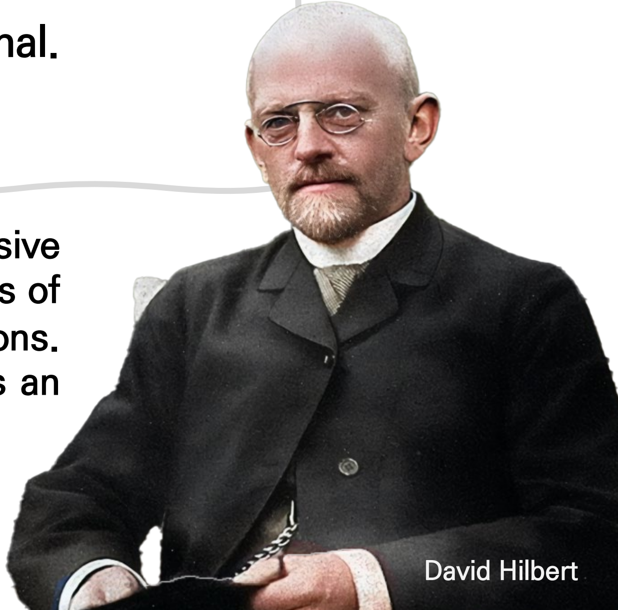
Choose $\mathrm{x} = \sqrt{2}$, $\mathrm{y} = \sqrt{2}$. Then $x^y = \sqrt{2}^{\sqrt{2}}$, which is assumed to be rational by hypothesis.

Case 2: $\sqrt{2}^{\sqrt{2}}$ is irrational.

Choose $x = \sqrt{2}^{\sqrt{2}}$, $\mathrm{y} = \sqrt{2}$. Then $x^y = (\sqrt{2}^{\sqrt{2}})^{\sqrt{2}} = \sqrt{2}^{\sqrt{2}\cdot\sqrt{2}} = \sqrt{2}^2 = 2$, which is clearly rational.

In either case, we conclude there exist real numbers x and y for which x^y is rational.

Note. This proof is often cited alongside the Gelfond–Schneider Theorem (1934), which provided a decisive solution to Hilbert's 7th Problem, posed by David Hilbert in 1900. It centers on whether expressions of the form a^b can be transcendental, where a and b are algebraic numbers under certain conditions. Specifically, Hilbert asked for a proof that if a is an algebraic number not equal to 0 or 1, and b is an irrational algebraic number, then a^b must be transcendental.



The Gelfond–Schneider Theorem(1934). If a and b are algebraic numbers with $\mathrm{a} \neq 0{,}1$ and b an irrational algebratic number, then any value of a^b is transcendental.

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$$
\dfrac{\begin{array}{c}[\ \sqrt{2}^{\sqrt{2}} \text{ is rational.}\ ] \\ \vdots \\ \text{Choose } x = \sqrt{2},\ y = \sqrt{2} \\ \vdots \\ \text{There are two real numbers } x \text{ and } y \\ \text{such that } x^y \text{ is rational.}\end{array} \qquad \begin{array}{c}[\ \sqrt{2}^{\sqrt{2}} \text{ is irrational.}\ ] \\ \vdots \\ \text{Choose } x = \sqrt{2}^{\sqrt{2}},\ y = \sqrt{2}. \\ \vdots \\ \text{There are two real numbers } x \text{ and } y \\ \text{such that } x^y \text{ is rational.}\end{array}}{\begin{array}{c}\text{There are two real numbers } x \text{ and } y \\ \text{such that } x^y \text{ is rational.}\end{array}}\ \text{Dil}
$$

$$
\dfrac{\begin{array}{c}[A] \\ \vdots \\ C\end{array} \qquad \begin{array}{c}[\neg A] \\ \vdots \\ C\end{array}}{C}\ \text{D}
$$

This proof is not constructive because it does not tell us whether $\sqrt{2}^{\sqrt{2}}$ is rational or irrational. In other words, a proof relying on the law of excluded middle only establishes the existence of something but cannot be considered constructive. To be constructive, a proof must explicitly demonstrate how a numerical entity is constructed or defined.

L.E.J. Brouwer Giving a Lecture
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Let us make the proof constructive.

Lemma. ${\sqrt{2}}^{\sqrt{2}}$ is irrational.

Proof. Let $z \neq 0$ and suppose $z = {\sqrt{2}}^{\sqrt{2}}$. Taking the base−2 logarithm of both sides gives

$$log_2 z = log_x \left({\sqrt{2}}^{\sqrt{2}}\right) = \frac{1}{\sqrt{2}}.$$

Therefore, $z = 2^{\frac{1}{\sqrt{2}}}$. Since $\frac{1}{\sqrt{2}}$ is an irrational exponent, it follows that z cannot be written as a ratio of two integers. Consequently, ${\sqrt{2}}^{\sqrt{2}}$ is irrational.

Proposition*. There exists two real numbers x and y such that x^y is rational.

Proof. Choose $x = {\sqrt{2}}^{\sqrt{2}}$ and $\mathrm{y} = \sqrt{2}$ that both are irrational.

Then $x^y = ({\sqrt{2}}^{\sqrt{2}})^{\sqrt{2}} = {\sqrt{2}}^{\sqrt{2}\cdot\sqrt{2}} = {\sqrt{2}}^2 = 2$, which is clearly rational.

Before examining Brouwer's counterexample to classical inferences, it is important to briefly mention a relevant historical point. Brouwer was not the first to express skepticism or raise concerns about the usefulness or validity of the law of excluded middle within the context of pure mathematics.

2. Some Histories

2.1. History of 'Constructive Proof': L.E.J. Brouwer

2.1.3. Precursors

Leopold Kronecker
1823. 12. 7 ~ 1891. 12. 29

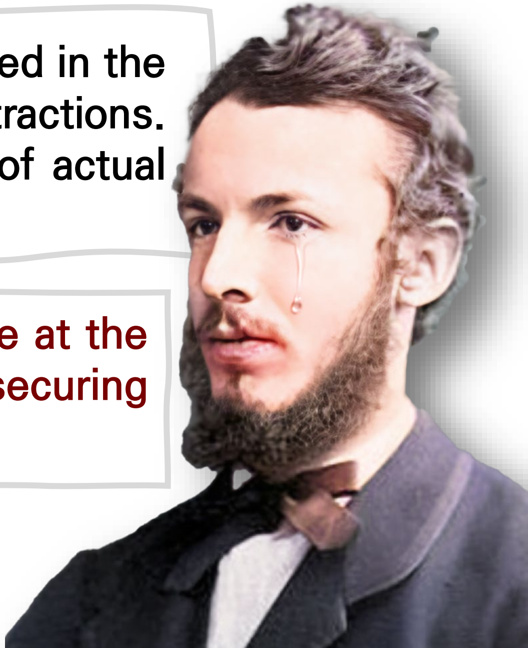
Leopold Kronecker was a prominent 19th-century European mathematician who made significant contributions to number theory and algebra. He spent the majority of his academic career at the University of Berlin, working closely with Ernst Kummer and eventually succeeding him as a professor in 1883.

"God made the integers, all else is the work of man."
(*Die ganzen Zahlen hat der liebe Gott gemacht, alles andere ist Menschenwerk*)

Weber, H. 1893. "Leopold Kronecker." *Mathematische Annalen* 43: 1–25.

This statement reflects his belief that mathematics should be grounded in the natural numbers, viewing other constructs as human-made abstractions. Kronecker's constructivist philosophy led him to reject the concept of actual infinity, opposing the treatment of infinite sets as completed entities.

Kronecker opposed Cantor's work on set theory and actual infinity, using his influence at the University of Berlin to block Cantor's papers, hinder his career, and prevent him from securing a professorship at the University of Berlin, highlighting their philosophical divide.



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From the 1870s, Kronecker objected to the unlimited use of the law of excluded middle and of definition by undecided separation of cases. For example, in his treatise on algebraic numbers of 1882, he wrote on the factorization of polynomial functions:

The definition of irreducibility drawn up in section 1 lacks a secure grounding as long as no method has been indicated by which it can be decided whether a definite given function is irreducible according to that definition or not.

He added in a footnote,

The analogous need, which as a matter of fact has often remained neglected, arises in many other cases, in definitions as in demonstrations, and on another occasion I will come back to this generally and thoroughly.

Kronecker, Leopold. 1882. ‘Grundzüge einer arithmetischen Theorie der algebraischen Größen’, Journal für die reine und angewandte Mathematik, 92, 1–122.

For reference, Jules Molk, a disciple of Kronecker, also gave voice to doubts about the law of the excluded middle.

2. Some Histories

2.1. History of 'Constructive Proof': L.E.J. Brouwer

2.1.4. Brouwer's Weak Counterexample to LEM

L.E.J. Brouwer Giving a Lecture
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Goldbach Conjecture. Every even number greater than or equal to 4 can be expressed as the sum of two prime numbers.

- Originally stated by Christian Goldbach in 1742, it is one of the oldest unsolved problems in number theory, requiring enormous computational checks to verify for large even numbers, yet a complete proof remains elusive.

Let $\alpha(n)$ be a sequence of rational numbers defined in terms of Goldbach's conjecture, as follows:

$$\alpha(n) = \begin{cases} -\left(\frac{1}{2}\right)^n & \text{if for all } j \leq n, 2j+4 \text{ is the sum of two primes} \\ -\left(\frac{1}{2}\right)^k & \text{if for some } k \leq n, 2k+4 \text{ is the not the sum of two primes} \end{cases}$$

If Goldbach's conjecture is true, the sequence behaves one way (it converges to 0).

If Goldbach's conjecture is false, it behaves differently (it converges to a non-zero value).

Because we do not know whether Goldbach's conjecture is true or false, we cannot (constructively) claim that $(\alpha = 0 \lor \alpha \neq 0)$.

Brouwer, L.E.J. "On the Significance of the Principle of Excluded Middle in Mathematics, Especially in Function Theory." In *Collected Works*, Vol. 1: *Philosophy and Foundations of Mathematics*, edited by A. Heyting, 334–345. Amsterdam: North-Holland Publishing Company, 1975.

Brouwer, L.E.J. "Mathematics, Science, and Language." In *Collected Works*, Vol. 1: *Philosophy and Foundations of Mathematics*, edited by A. Heyting, 477–484. Amsterdam: North-Holland Publishing Company, 1975.

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The sequence of the $\alpha(n)$ satisfies the Cauchy condition (the condition that for every rational number $\varepsilon > 0$ there is a natural number N such that $|\alpha(j) - \alpha(k)| < \varepsilon$ for all $j, k > \mathrm{N}$), as for every n, any two members of the sequence after $\alpha(n)$ lie within $\left(\frac{1}{2}\right)^n$ of each other. Therefore the sequence converges and determines a real number α.

This leads to the following three weak counterexamples:

1. We cannot now assert $\forall x \in \mathbb{R}\,(x = 0 \lor x \neq 0)$, because we cannot, intuitionistically, assert $(\alpha = 0 \lor \alpha \neq 0)$ until we have a proof of one of the disjuncts.
2. Similarly, we cannot now assert $\forall x \in \mathbb{R}\,(x < 0 \lor x = 0 \lor x > 0)$.
3. We cannot now assert $\forall x \in \mathbb{R}\,(x \in \mathbb{Q} \lor x \notin \mathbb{Q})$, for to assert that $\alpha \in \mathbb{Q}$ we have to know $m, n \in \mathbb{Z}$ such that $\alpha = \frac{m}{n}$, but we can't as long as we do not know the value of α. (By construction, α cannot be irrational.)

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2.1.4. Brouwer’s Weak Counterexample to LEM

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Brouwer’s proposal was not an example demonstrating the falsity of the Law of Excluded Middle, but rather an example showing that we may lack recognizable proofs for both a sentence and its negation.

2. Some Histories

2.1. History of 'Constructive Proof': L.E.J. Brouwer

2.1.5. Brouwer's Strong Counterexample to LEM

L.E.J. Brouwer Giving a Lecture
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Theorem (Brouwer 1928). Let $\mathbb{R}$ be an intuitionistic continuum, and let Px denote the predicate "x is rational." Then Brouwer's argument establishes that $\neg\forall x \in \mathbb{R}\,(Px \lor \neg Px)$.

Brouwer, Luitzen Egbertus Jan. 1927. "On the Domains of Definition of Functions." *KNAW Verhandelingen* 31: 1–87.
Brouwer, Luitzen Egbertus Jan. 1928. "Reflections on Formalism." *KNAW Proceedings* 31: 374–379.
van Atten, Mark , "Luitzen Egbertus Jan Brouwer." *Stanford Encyclopedia of Philosophy*.

In other words, there is no intuitionistically valid proof of the classical law of excluded middle for the property P over $\mathbb{R}$.

Weak Counterexample: it is a counterexample to a statement φ occurs when φ is unprovable in a constructive or intuitionistic system, but not formally refuted. In other words, the system remains undecided on φ's truth precisely because φ depends on an open or unresolved problem (e.g., Goldbach's conjecture).

Strong Counterexample: it is a counterexample to a statement φ arises when the system does in fact prove $\neg\varphi$ or shows that assuming φ leads to a contradiction with core constructive principles.

2. Some Histories

2.1. History of 'Constructive Proof': L.E.J. Brouwer

2.1.6. On Brouwer's Negations

L.E.J. Brouwer Giving a Lecture
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Weak Counterexample: it is a counterexample to a statement φ occurs when φ is unprovable in a constructive or intuitionistic system, but not formally refuted.

Strong Counterexample: it is a counterexample to a statement φ arises when the system does in fact prove $\neg\varphi$ or shows that assuming φ leads to a contradiction with core constructive principles.

Two counterexamples to LEM says: "LEM is not true."

1. "LEM is not true" in a weak sense. (not proven)

- We *have not established* that the law of excluded middle (LEM) holds universally in intuitionistic mathematics. At present, we cannot prove LEM for all statements within our constructive setting. However, we also have not shown that LEM is outright false—so far, there is no contradiction.

2. "LEM is not true" in a strong sense. (proven false, disproof)

- LEM is *actually refuted* in certain intuitionistic frameworks. Concretely, one can prove that adopting LEM leads to a contradiction with core constructive principles, forcing us to reject LEM entirely within intuitionistic mathematics.

2. Some Histories

2.2. History of 'Constructive Proof': Michael Dummett

2.2.1. Michael Dummett: Frege meets Wittgenstein

Michael Dummett
1925.6.27 – 2011.12.27

Major Contributions

1. Philosophy of Language: Michael Dummett developed anti-realism, arguing that a statement's meaning depends on its verifiability rather than its alignment with an objective reality. He also offered groundbreaking interpretations of Gottlob Frege's work, shaping modern analytic philosophy.

2. Logic and Mathematics: Dummett championed intuitionistic logic, rejecting the classical law of the excluded middle and emphasizing constructive proofs to establish truth.

3. Influential Works: *Frege: Philosophy of Language* (1973), *Truth and Other Enigmas* (1978), *The Logical Basis of Metaphysics* (1991)

Social Justice Activism and Legacy

Michael Dummett was a passionate advocate for racial equality and immigrant rights. He co-founded the Institute of Race Relations and the Joint Council for the Welfare of Immigrants (JCWI), promoting open borders and challenging racial discrimination in immigration policies. His contributions earned him the Lakatos Award (1994), the Rolf Schock Prize (1995), and a knighthood in 1999. Dummett's legacy encompasses both his transformative impact on analytic philosophy and his enduring influence on global movements for social justice and equality.

Dummett with his wife Ann and their five children; Ann worked alongside him in their fight against racism.

2. Some Histories

2.2. History of 'Constructive Proof': Michael Dummett

2.2.1. Michael Dummett: Frege meets Wittgenstein

Immanuel Kant
1724.4.22. – 1804.2.12

How are numbers given to us?

Numbers are given to us through our fundamental intuition of time and our active mental construction based on this intuition. They are not pre-existing logical objects but are constructed through our mathematical activity grounded in temporal consciousness. That is, Mathematics is a free creation of the human mind and logic is not an absolutely reliable instrument to discover mathematical truths.

L. E. J. Brouwer
1881–1966
인싸 of 인싸

No. Such a view merely reduces mathematics to a subjective, psychological product. We can investigate mathematical truths by examining pure number concepts independent of such subjective psychology, and it is logic that makes this investigation possible. In fact, I maintain that mathematics is logic.

Gottlob Frege
1848.11.8. – 1925.7.26
아싸 of 아싸

Caution: It should be noted that this exchange is a philosophical reconstruction and not an actual historical dialogue between Brouwer and Frege.

Frege regarded psychology as inherently subjective. While logic was considered a branch of psychology at the time, Frege insisted that logic was not psychological but rather a fundamental tool for investigating objective truth. This conviction ultimately led him to his groundbreaking development of modern predicate logic.

2. Some Histories

2.2. History of 'Constructive Proof': Michael Dummett

2.2.1. Michael Dummett: Frege meets Wittgenstein

Gottlob Frege
1848.11.8. – 1925.7.26

- Frege developed logic from what was considered psychology into modern predicate logic with his first book, *Begriffsschrift*, published in 1879.

1. *Begriffsschrift* : the creation of predicate logic

Frege laid the groundwork for modern predicate logic by analyzing sentences in a way that distinguished between names and predicates, rather than the traditional way of analyzing sentences around subjects and predicates.

'Begriffsschrift,' literally 'concept notation,' represents Frege's attempt to create a rigorous, formal symbolic language that mirrors the logical structure of thought.

"If it is one of the tasks of philosophy to break the domination of the word over the human spirit by laying bare the misconceptions that through the use of language often almost unavoidably arise concerning the relations between concepts and by freeing thought from that with which only the means of expression of ordinary language, constituted as they are, saddle it, then my ideography, further developed for these purposes, can become a useful tool for the philosopher."

G. Frege, *Beggriffschrift*, 1879, pp. 6–7.

Reference. The year **1879** marked two significant milestones: the publication of Begriffsschrift, which asserted that logic was distinct from subjective psychology, and Wilhelm Wundt's establishment of the world's first psychological laboratory at the University of Leipzig, aimed at advancing psychology as an objective experimental science.

Wundt's psychological laboratory

2. Some Histories

2.2. History of ‘Constructive Proof’: Michael Dummett

2.2.1. Michael Dummett: Frege meets Wittgenstein

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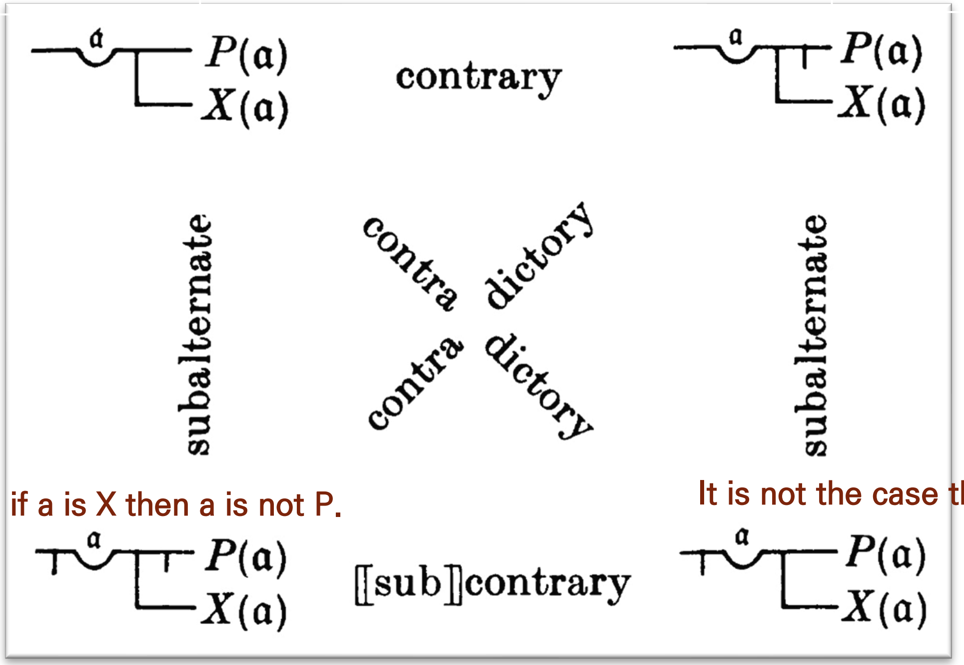
1. *Begriffsschrift* : the creation of predicate logic

Frege laid the groundwork for modern predicate logic by analyzing sentences in a way that distinguished between names and predicates, rather than the traditional way of analyzing sentences around subjects and predicates.

For every a, if a is X then a is P.

For every a, if a is X then a is not P.



It is not the case that, for every a, if a is X then a is not P.
(For some a, a is X and a is P.)

It is not the case that, for every a, if a is X then a is P.
(For some a, a is X and a is not P.)

2. Some Histories

2.2. History of ‘Constructive Proof’: Michael Dummett

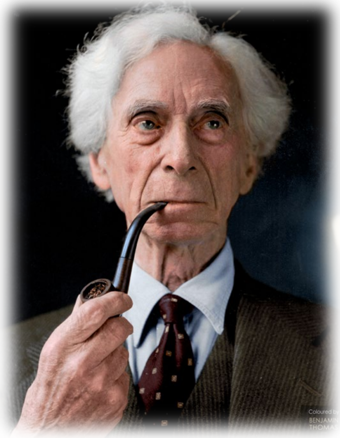
2.2.1. Michael Dummett: Frege meets Wittgenstein

Gottlob Frege
1848.11.8. – 1925.7.26

- Frege wrote *Die Grundlagen der Arithmetik* with the belief that mathematics is reducible to logic.

2. *Die Grundlagen der Arithmetik*

Frege then went on to write *Die Grundlagen der Arithmetik* to show that logical and mathematical truths are analytical truths, characterized as ones that are true by virtue of the meanings of their words alone and/or can be known to be so solely by knowing those meanings.



Bertrand Russell
1872. 5. 18. ~ 1970. 2. 2.

Russell identified a critical flaw in Frege's arithmetic system: it could express paradoxical concepts such as "the set of all elements that do not contain themselves." Russell demonstrated that permitting such expressions would lead to inconsistencies within Frege's system. Reportedly, Frege was so devastated by this revelation, viewing it as the collapse of his life's work, that he withdrew from publishing any papers for approximately ten years.

Russell, Bertrand, 1902. “Letter to Frege,” in Jean van Heijenoort (ed.), *From Frege to Gödel*, Cambridge, Mass.: Harvard University Press, 1967, 124–125.

Frege, Gottlob, 1902. “Letter to Russell,” in Jean van Heijenoort (ed.), *From Frege to Gödel*, Cambridge, Mass.: Harvard University Press, 1967, 127–128.

The scholarly significance of Frege's *Die Grundlagen der Arithmetik* was (most likely first) recognized by Russell – the same person who would later dismantle Frege's life's work.

2. Some Histories

2.2. History of 'Constructive Proof': Michael Dummett

2.2.1. Michael Dummett: Frege meets Wittgenstein

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2. *Die Grundlagen der Arithmetik*

Frege then went on to write *Die Grundlagen der Arithmetik* to show that logical and mathematical truths are analytical truths, characterized as ones that are true by virtue of the meanings of their words alone and/or can be known to be so solely by knowing those meanings.

How ... are numbers to be given to us, if we cannot have any ideas or intuition of them? Since it is only in the context of a proposition that words have any meaning, our problem becomes this: To define the sense of a proposition in which a number word occurs.

G. Frege, *Die Grundlagen der Arithmetik*, Sec. 62.

His solution was to invoke the context principle: only in the context of a sentence does a word have meaning. On the strength of this, Frege converts the problem into an enquiry how the senses of sentences containing terms for numbers are to be fixed. There is the linguistic turn. The context principle is stated as an explicitly linguistic one, a principle concerning the meanings of words and their occurrence in sentences; and so an epistemological problem, with ontological overtones, is by its means converted into one about the meanings of sentences.

M. Dummett, *Frege: Philosophy of Mathematics*, p. 111

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2. Some Histories

2.2. History of ‘Constructive Proof’: Michael Dummett

2.2.1. Michael Dummett: Frege meets Wittgenstein

Ludwig Wittgenstein
1889.4.26 – 1951.4.29

Early Wittgenstein (*Tractatus Logico-Philosophicus*):

Wittgenstein aimed to demonstrate that language, thought, and reality share a common logical structure. Through the Tractatus, he sought to show that meaningful language must picture facts in the world, while metaphysical statements are nonsensical. He argued that philosophy's role is to clarify logical form and draw limits to what can be said, ultimately revealing that the most important things in life cannot be expressed but only shown. The work culminates in the famous assertion that “whereof one cannot speak, thereof one must be silent.”

Later Wittgenstein (Philosophical Investigations):

In his later work, Wittgenstein radically shifted to viewing language as a collection of context-dependent “language games” rather than a rigid logical structure. He aimed to show that philosophical problems arise from misunderstanding the everyday use of language, emphasizing that meaning comes from use within specific contexts or “forms of life.” Through detailed examples and observations, he demonstrated that traditional philosophical questions often result from linguistic confusion. His goal was therapeutic: to help philosophers stop being trapped by their own misuse of language and return words to their ordinary usage.

2. Some Histories

2.2. History of 'Constructive Proof': Michael Dummett

2.2.1. Michael Dummett: Frege meets Wittgenstein

Ludwig Wittgenstein
1889.4.26 – 1951.4.29

4.024 To understand a proposition means to know what is the case if it is true. (One can understand it. Therefore, without knowing whether it is true.) It is understood by anyone who understands its constituents.

L. Wittgenstein, *Tractatus Logico Philosophicus*

43. For a large class of cases—though not for all—in which we employ the word "meaning" it can be defined thus: the meaning of a word is its use in the language.

L. Wittgenstein, *Philosophical Investigations*

Verificationist Thesis: To understand the meaning of a statement is to understand under what conditions it would be true. (The meaning of a statement is its verification conditions.)

Meaning-Use Thesis: The meaning of a statement is determined entirely by its use in language.

I got the verification thesis from *Tractatus* and the meaning-use thesis from *Philosophical Investigations*.

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2. Some Histories

2.2. History of 'Constructive Proof': Michael Dummett

2.2.1. Michael Dummett: Frege meets Wittgenstein

Michael Dummett
1925.6.27 – 2011.12.27

Three Theses of Dummett's Theory of Meaning

Context Principle: Words have meaning only within the context of a sentence.

Meaning-Use Thesis: The meaning of a statement is determined entirely by its use in language.

Verificationist Thesis: To understand the meaning of a statement is to understand under what conditions it would be true. (The meaning of a statement is its verification conditions.)

Dummett marshals three fundamental principles – the context principle, meaning-use thesis, and verification thesis – to mount a comprehensive critique of semantic realism. This critique fundamentally entails a rejection of the principle of bivalence.

Semantic Realism: every declarative sentence of one's language is determinately true or false, independently of our means of coming to know which.

Principle of Bivalence: For every declarative sentence A, A is true or false.

As an intuitionist, Dummett denies that the following is a logical truth:

"Either every even number greater than 2 is the sum of two primes, or not every even number greater than 2 is the sum of two primes." (Either Goldbach's conjecture holds, or it does not hold.)

2. Some Histories

2.2. History of 'Constructive Proof': Michael Dummett

2.2.2. Dummett's Argument Against Semantic Realism


Michael Dummett Giving a Lecture Generated by Midjourney

Context Principle: Words have meaning only within the context of a sentence.

Meaning-Use Thesis: The meaning of a statement is determined entirely by its use in language.

Communicability Thesis. The meaning of a statement consists solely in its role as an instrument of communication between individuals.

So far as I am able to see, there are just two lines of argument for repudiating classical reasoning in mathematics in favour of intuitionistic reasoning. The first runs along the following lines. The meaning of a mathematical statement determines and is exhaustively determined by its use. The meaning of such a statement cannot be, or contain as an ingredient, anything which is not manifest in the use made of it, lying solely in the mind of the individual who apprehends that meaning: if two individuals agree completely about the use to be made of the statement, then they agree about its meaning. The reason is that the meaning of a statement consists solely in its rôle as an instrument of communication between individuals, just as the powers of a chess-piece consist solely in its rôle in the game according to the rules. An individual cannot communicate what he cannot be observed to communicate: if one individual associated with a mathematical symbol or formula some mental content, where the association did not lie in the use he made of the symbol or formula, then he could not convey that content by means of the symbol or formula, for his audience would be unaware of the association and would have no means of becoming aware of it.

M. Dummett (1973), "The Philosophical Basis of Intuitionistic Logic," *Truth and Other Enigmas*, pp. 216–217.

Manifestation Argument

2. Some Histories

2.2. History of 'Constructive Proof': Michael Dummett

2.2.2. Dummett's Argument Against Semantic Realism

Michael Dummett
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Context Principle: Words have meaning only within the context of a sentence.

Meaning-Use Thesis: The meaning of a statement is determined entirely by its use in language.

Communicability Thesis. The meaning of a statement consists solely in its role as an instrument of communication between individuals.

Manifestation Argument: There no meaning of sentence incapable of being manifested in individual's linguistic use.

(1) If there exists a meaning of sentence which is unable to be manifested in individual's linguistic use, then the knowledge of the sentence cannot be manifested in individual's linguistic use.
(2) If the knowledge of the sentence cannot be manifested in individual's linguistic use, then the knowledge of the sentence cannot be communicated by the individuals.
(3) If the knowledge of the sentence cannot be communicated by the individuals, then, by the Communicability Thesis, the knowledge of the sentence is not anymore meaning.
(4) This is a contradiction. Therefore, there does not exist a statement which is unable to be manifested in individual's linguistic use.

2. Some Histories

2.2. History of 'Constructive Proof': Michael Dummett

2.2.2. Dummett's Argument Against Semantic Realism

Michael Dummett
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Context Principle: Words have meaning only within the context of a sentence.

Meaning-Use Thesis: The meaning of a statement is determined entirely by its use in language.

Communicability Thesis. The meaning of a statement consists solely in its role as an instrument of communication between individuals.

Verificationist Thesis: To understand the meaning of a statement is to understand under what conditions it would be true. (The meaning of a statement is its verification conditions.)

Manifestation Argument: There is no meaning of sentence incapable of being manifested in individual's linguistic use.

Fact 1. There is no known proof or disproof of the Goldbach Conjecture as of January 2025.

Argument for the Extra-Linguistic Entity (AELE): If the Principle of Bivalence is true, then there exists a statement with an unknowable, non-linguistic meaning.

1. If the Principle of Bivalence is true, then every statement must have either a proof or a disproof.
2. If Fact 1 holds true and the Principle of Bivalence is valid, there must exist a proof or disproof for certain statements that are unknowable.
3. If such unknowable proofs or disproofs exist for a statement, then there exist verification conditions for that statement that are unknowable.
4. If there are verification conditions for a statement that are unknowable, then there must also exist an unknowable, non-linguistic meaning for that statement (based on the Communication Thesis and the Verificationist Thesis).

2. Some Histories

2.2. History of 'Constructive Proof': Michael Dummett

2.2.2. Dummett's Argument Against Semantic Realism


Michael Dummett Giving a Lecture Generated by Midjourney

Context Principle: Words have meaning only within the context of a sentence.

Meaning-Use Thesis: The meaning of a statement is determined entirely by its use in language.

Communicability Thesis. The meaning of a statement consists solely in its role as an instrument of communication between individuals.

Verificationist Thesis: To understand the meaning of a statement is to understand under what conditions it would be true. (The meaning of a statement is its verification conditions.)

Manifestation Argument: There is no meaning of sentence incapable of being manifested in individual's linguistic use.

Argument for the Extra-Linguistic Entity (AELE): If the Principle of Bivalence is true, then there exists a statement with an unknowable, non-linguistic meaning.

Argument for the Rejection of the Principle of Bivalence: the Principle of Bivalence is not true.

Assume: the Principle of Bivalence is true.

1. (By AELE) There exists a statement with an unknowable, non-linguistic meaning.
2. If there exists a statement with an unknowable, non-linguistic meaning, then the meaning of that statement cannot be manifested in individual language use. (Contradiction with the Manifestation Argument)

So, the SEMANTIC REALISM is false.

2. Some Histories

2.2. History of 'Constructive Proof': Michael Dummett

2.2.3. Dummett's Intuitionism (Anti-Realist Constructive Logicism)

Michael Dummett
1925.6.27 – 2011.12.27

It is widely acknowledged among philosophers of language that Dummett first distinguished semantic realism from semantic anti-realism in a systematic, philosophically precise manner. The debate it ignited remains central in contemporary discussions about the nature of meaning, truth, and verification.

Semantic Realism: every (meaningful) statement has a determinate truth value – true or false – regardless of whether we, as humans, can effectively recognize or verify that truth value. In other words, a statement's truth conditions exist independently of our cognitive or epistemic capacities; there is something "in the world" making it the case that the statement is true or false, even if we lack a feasible method for deciding which.

Semantic Anti-Realism: Every truth is knowable. If we have no principled means—now or in a conceivable future—to establish whether a statement is true or false, then we are not warranted in claiming that there is a fact of the matter that makes it definitively one or the other.

The realist holds that we give sense to those sentences of our language which are not effectively decidable by appealing tacitly to means of determining their truth-values which we do not our-selves possess, but which we can conceive of by analogy with those which we do. The anti-realist holds that such a conception is quite spurious, an illusion of meaning, and that the only meaning we can confer on our sentences must relate to those means of determining their truth-values which we actually possess. Hence, unless we have a means which would in principle decide the truth-value of a given statement, we do not have for it a notion of truth and falsity which would entitle us to say that it must be either true or false.

M. Dummett (1959), "Truth," *Truth and Other Enigmas*, pp. 24.

2. Some Histories

2.2. History of 'Constructive Proof': Michael Dummett

2.2.3. Dummett's Intuitionism (Anti-Realist Constructive Logicism)

Michael Dummett
1925.6.27 – 2011.12.27

Semantic Realism: every declarative sentence of one's language is determinately true or false, independently of our means of coming to know which. (Not every truth is knowable.)
Semantic Anti-Realism: Semantic realism is false. (Every truth is knowable.)

Dummett's philosophical contribution extends beyond Brouwer's mathematical counterexamples to the Law of Excluded Middle (LEM). His crucial insight was twofold:

1. He revealed that semantic realism serves as the foundational justification for LEM.
2. He attempted to demonstrate that the problems with semantic realism extend beyond mathematics into the domain of ordinary language.

Based on this comprehensive critique, Dummett argues for a fundamental shift in the theory of meaning, proposing that proof, not truth, should serve as its central concept.

We must ... replace the notion of truth, as the central notion of the theory of meaning for mathematical statements, by the notion of *proof*: a grasp of the meaning of a statement consists in a capacity to recognize a proof of it when one is presented to us, and a grasp of the meaning of any expression smaller than a sentence must consist in a knowledge of the way in which its presence in a sentence contributes to determining what is to count as a proof of that sentence.

M. Dummett (1973), "The Philosophical Basis of Intuitionistic Logic," *Truth and Other Enigmas*, pp. 225–226.

2. Some Histories

2.2. History of ‘Constructive Proof’: Michael Dummett

2.2.3. Dummett’s Intuitionism (Anti-Realist Constructive Logicism)

Michael Dummett
1925.6.27 – 2011.12.27

Semantic Realism: every declarative sentence of one’s language is determinately true or false, independently of our means of coming to know which. (Not every truth is knowable.)

Semantic Anti-Realism: Semantic realism is false. (Every truth is knowable.)

Frege

Frege famously advanced logicism, the view that mathematics is reducible to logic, operating within a classical framework that accepts the law of excluded middle.

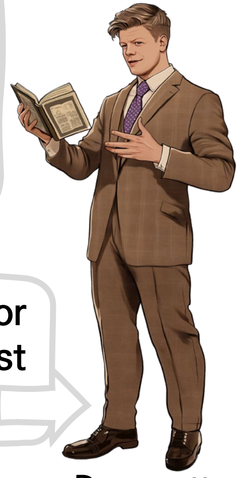
Wittgenstein

Dummett, influenced by Wittgenstein’s emphasis on the use of language, revisited Frege’s logicism from an ordinary language perspective, placing special importance on how meaning is established through verification practices.

Brouwer

Aware of the leading positions in the philosophy of mathematics—particularly intuitionism—Dummett championed a version of intuitionistic logic for both formal mathematics and everyday language. This approach drew on his manifestation argument, which holds that understanding a statement requires being able to show how one would recognize its truth or falsity.

Consequently, Dummett’s position came to be seen as a form of anti-realist (or constructive) logicism, integrating Frege’s core insights with a constructive, anti-realist commitment that departs from strictly classical logic.



Dummett

2. Some Histories

2.2. History of 'Constructive Proof': Michael Dummett

2.2.4. After Dummett ...

Neil Tennant
1950.3.1 – Present

Intuitionistic Relevant Logic (Core Logic)

The following principle is a cornerstone of prooftheoretic foundations for constructive mathematics:

For every proof Π that we may provide for a mathematical theorem φ, it must be possible, in principle, to transform Π, *via* a finite sequence of applicable reduction procedures, into a *canonical* proof of φ, that is, a proof of φ that is *in normal form*, so that none of the reduction procedures is applicable to it.

Neil Tennant, "A New Unified Account of Truth and Paradox," *Mind*, Vol. 124.

Proof-Theoretic Semantics

Gerhard Gentzen
1909.11.4 – 1945.8.4

Dag Prawitz
1936.5.16 – Present

Peter Schroeder-Heister
1953.3.2 – Present

Neo Logicism

Crispin Wright
1942.12.21 – Present

2. Some Histories

2.2. History of 'Constructive Proof': Michael Dummett

2.2.5. Dummett's Negations

Michael Dummett
(1925.6.27 – 2011.12.27)

Weak Counterexample: it is a counterexample to a statement φ occurs when φ is unprovable in a constructive or intuitionistic system, but not formally refuted.

Strong Counterexample: it is a counterexample to a statement φ arises when the system does in fact prove $\neg\varphi$ or shows that assuming φ leads to a contradiction with core constructive principles.

Brouwer

[W]e may call a 'recognisably strong counter-example' to this rule of inference, namely, a putative inference exemplifying the rule in question whose premisses are recognisably true and whose conclusion is recognisably false. ⋯ A recognisably weak counter-example will be one in which the premisses are recog nisably true but the conclusion recognisably not true, but not recognis ably false.

M. Dummett, *Logical Basis of Metaphysics*, Harvard University Press, p. 189.

Two counterexamples to LEM says: "LEM is not true."

1. "LEM is not true" in a weak sense. (not proven)
2. "LEM is not true" in a strong sense. (proven false, disproof)

2. Some Histories

2.2. History of 'Constructive Proof': Michael Dummett

2.2.5. Dummett's Negations

Michael Dummett
(1925.6.27 – 2011.12.27)

Verificationist Thesis: To understand the meaning of a statement is to understand under what conditions it would be true. (The meaning of a statement is its verification conditions.)

For any sentence A:

1. A is true if and only if there exists a constructive proof of A.
2. A is false if and only if there exists a constructive disproof of A.

Brouwer

Two counterexamples to LEM says: "LEM is not true."

1. "LEM is not true" in a weak sense. (not proven)
2. "LEM is not true" in a strong sense. (proven false, disproof)

Two senses of "A is NOT true."

1'. It is not the case that there exists a constructive proof of A. (not proven)
   – No one has yet provided a constructive proof of A.

2'. There exists a constructive disproof of A. (proven false)

3. Two Intuitionistic Senses of 'Not True'

3.1. Three Characteristics of Classical Logic

3.1.1. Intuitionists' Rejection of the Thesis 2

Thesis 1. There are only two truth values: true and false.
Thesis 2. No sentence is neither true nor false.
Thesis 3. No sentence is both true and false.

A	(A∨¬A)	(A∧¬A)	¬(A∧¬A)	¬(A∨¬A)
T	T	F	T	F
F	T	F	T	F

Dummett

Intuitionists reject Thesis 2. They argue that some sentences may lack a truth value if their constructive proof is unknown.

However, it is important to note that intuitionists do not reject Theses 1 and 3. They do not assert the existence of a third truth value. Furthermore, rejecting the law of excluded middle is not equivalent to claiming that A∨¬A has a disproof, which which would imply that A∧¬A has a proof.

But, notoriously, the anti-realist has to be very careful indeed not to express her refusal to accept Bivalence in the form of a claim to the effect that there are counterexamples to it. Rather, she should go no further than to assert $\neg\forall\varphi\,(T[\varphi] \vee \neg T[\varphi])$. The anti-realist, who does not accept full classical logic, does not allow that this entails $\exists\varphi\,\neg(T[\varphi] \vee \neg T[\varphi])$. And it is important to avoid commitment to the latter, since it is intuitionistically inconsistent, as the following lemma makes clear.

Tennant

Neil Tennant, "A New Unified Account of Truth and Paradox," *Mind*, Vol. 124, p. 582.

3. Two Intuitionistic Senses of ‘Not True’

3.2. Intuitionistic Negations

3.2.1. ‘Not True’ and ‘False’

Verificationist Thesis: To understand the meaning of a statement is to understand under what conditions it would be true.

For any sentence A:

1. A is true if and only if there exists a constructive proof of A.
2. A is false if and only if there exists a constructive disproof of A.

In intuitionism, “A is not true” and “A is false” are not always equivalent.

A is not true. (¬A)

(1) It is not the case that there exists a constructive proof that A is true. (not proven)
– No one has yet provided a constructive proof that A is true.

(2) There exists a constructive disproof that A is true. (proven false)

Note that (2) Implies (1) but not vice versa.

How about to the case of ¬¬A?

3. Two Intuitionistic Senses of ‘Not True’

3.2. Intuitionistic Negations

3.2.2. On ‘Not Not True’

Verificationist Thesis: To understand the meaning of a statement is to understand under what conditions it would be true.

For any sentence A:

1. A is true if and only if there exists a constructive proof that A is true.
2. A is false if and only if there exists a constructive disproof that A is false.

It is not the case that A is not true. ($\neg\neg A$)

Case 1. there exists a constructive disproof that A is true. (proven false, $\neg A$ has a proof)

(1’) No one has yet provided a constructive proof that there exists a constructive disproof that A is true.

(2’) there exists a constructive disproof that there exists a constructive disproof that A is true.

Case 2. No one has yet provided a constructive proof that A is true. (not proven, $\neg A$ has no proof yet.)

$\neg\neg A$ does not always imply A in intuitionistic interpretation of negation.

"A is not true." $\Leftrightarrow$ "There does not exist a constructive proof that A is true."

$\not\Rightarrow$ "There exists a constructive disproof that A is false."

"A is false." $\Leftrightarrow$ "There exists a constructive disproof that A is false."

$\Rightarrow$ "There does not exist a constructive proof that A is true."

$\Rightarrow$ "A is not true."

4. Formalizing Negations in Natural Deduction

4.1. ‘Not True’ in Natural Deduction

4.1.1. Two Problems of Intuitionistic Negations in Natural Deduction

Let T(x) be a predicate stating that x is true. “Φ is not true” is not equivalent to “not Φ is true”. However, in the standard natural deduction, it is proved that they are equivalent.

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{[\neg T(\ulcorner\Phi\urcorner)]^1 \quad \dfrac{[\Phi]^2}{T(\ulcorner\Phi\urcorner)}\,TI}{\bot}\,\to E}{\neg\Phi}\,\to I_{,2}}{T(\ulcorner\neg\Phi\urcorner)}\,TI}{\neg T(\ulcorner\Phi\urcorner)\to T(\ulcorner\neg\Phi\urcorner)}\,\to I_{,1} \qquad \dfrac{\dfrac{\dfrac{\dfrac{[T(\ulcorner\Phi\urcorner)]^3}{\Phi}\,TE \quad \dfrac{[T(\ulcorner\neg\Phi\urcorner)]^4}{\neg\Phi}\,TE}{\bot}\,\to E}{\neg T(\ulcorner\Phi\urcorner)}\,\to I_{,3}}{T(\ulcorner\neg\Phi\urcorner)\to\neg T(\ulcorner\Phi\urcorner)}\,\to I_{,4}}{\dfrac{(\neg T(\ulcorner\Phi\urcorner)\to T(\ulcorner\neg\Phi\urcorner))\wedge(T(\ulcorner\neg\Phi\urcorner)\to\neg T(\ulcorner\Phi\urcorner))}{\neg T(\ulcorner\Phi\urcorner)\leftrightarrow T(\ulcorner\neg\Phi\urcorner)}\,def}\,\wedge I
$$

Choi, S. (2018), “Liar-Type Paradoxes and Intuitionistic Natural Deduction Systems”, *Korean Journal of logic*, Vol. 21(1), pp. 59 – 96.

4. Formalizing Negations in Natural Deduction

4.1. ‘Not True’ in Natural Deduction

4.1.1. Two Problems of Intuitionistic Negations in Natural Deduction

Let T(x) be a predicate stating that x is true.

Tennant

But, notoriously, the anti-realist has to be very careful indeed not to express her refusal to accept Bivalence in the form of a claim to the effect that there are counterexamples to it. Rather, she should go no further than to assert $\neg\forall\varphi\,(T[\varphi] \vee \neg T[\varphi])$. The anti-realist, who does not accept full classical logic, does not allow that this entails $\exists\varphi\,\neg(T[\varphi] \vee \neg T[\varphi])$. And it is important to avoid commitment to the latter, since it is intuitionistically inconsistent, as the following lemma makes clear.

Neil Tennant, “A New Unified Account of Truth and Paradox,” *Mind*, Vol. 124, p. 582.

Lemma 2

$\neg \mathrm{D}\ell$, i.e. $\neg(\mathrm{T}\ell \vee \neg\mathrm{T}\ell)$

Proof (in normal form):

$$
\begin{prooftree}
\AxiomC{}
\LeftLabel{(2)}
\UnaryInfC{$\mathrm{T}\ell \vee \neg\mathrm{T}\ell$}
\AxiomC{}
\LeftLabel{(1)}
\UnaryInfC{$\mathrm{T}\ell$}
\AxiomC{}
\UnaryInfC{$\ell = \ulcorner \neg\mathrm{T}\ell \urcorner$}
\BinaryInfC{$\mathrm{T}\ulcorner \neg\mathrm{T}\ell \urcorner$}
\UnaryInfC{$\neg\mathrm{T}\ell$}
\AxiomC{}
\RightLabel{(1)}
\UnaryInfC{$\mathrm{T}\ell$}
\BinaryInfC{$\bot$}
\AxiomC{}
\RightLabel{(1)}
\UnaryInfC{$\neg\mathrm{T}\ell$}
\UnaryInfC{$\mathrm{T}\ulcorner \neg\mathrm{T}\ell \urcorner$}
\AxiomC{}
\UnaryInfC{$\ell = \ulcorner \neg\mathrm{T}\ell \urcorner$}
\BinaryInfC{$\mathrm{T}\ell$}
\AxiomC{}
\RightLabel{(1)}
\UnaryInfC{$\neg\mathrm{T}\ell$}
\BinaryInfC{$\bot$}
\RightLabel{(1)}
\TrinaryInfC{$\bot$}
\RightLabel{(2)}
\UnaryInfC{$\neg(\mathrm{T}\ell \vee \neg\mathrm{T}\ell)$}
\end{prooftree}
$$

Neil Tennant, “A New Unified Account of Truth and Paradox,” *Mind*, Vol. 124, p. 584.

4. Formalizing Negations in Natural Deduction

4.1. ‘Not True’ in Natural Deduction

4.1.1. Two Problems of Intuitionistic Negations in Natural Deduction

Two examples show that intuitionistic negations are not well formalized in the standard natural deduction system.

11.2 Excluded middle plays no essential role in the derivation of the paradoxes

The following proofs are all in Core Logic. Note that we are ‘lapsing’ into a use of the Gentzen–Prawitz proof system, which employs the *serial* forms of the Elimination rules for $\rightarrow$ and $\leftrightarrow$. This is because of the dialectical consideration that we wish to show that certain proofs *were to be had*, if only care had been taken to find them. The Gentzen–Prawitz system of natural deduction would have been an obvious early choice for the sorts of proof search that philosophical logicians needed to carry out before discussing the paradoxes and venturing to blame them on excluded middle. (Sometimes a public defender can be rightly convinced that an otherwise scurrilous defendant is not guilty as charged.)

Tennant (2018), *Core Logic*

Thank You for Listening!!

Happy Logic Day!!