On proof-theoretic dilator and Pohlers' characteristic ordinals

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Strength of theories

Gödel's incompleteness theorem shows no recursive theory interpreting arithmetic can prove its own consistency unless it is inconsistent.

A theory proving its own consistency	$\Longrightarrow$	A hierarchy of theories

Problem: How do we 'line up' theories into a hierarchy?

How to compare the strength of theories?

The most simple way to compare theories is inclusion. (Which theory proves more?)

Example

Clearly $\mathsf{ZFC} \subseteq \mathsf{ZFC} + \neg\mathsf{CH}$
However, a forcing argument shows if ZFC is consistent, then so is $\mathsf{ZFC} + \neg\mathsf{CH}$.

Hence both ZFC and $\mathsf{ZFC} + \neg\mathsf{CH}$ have the same "consistency strength."

Consistency strength

What happens if we compare theories by their "consistency strength?"

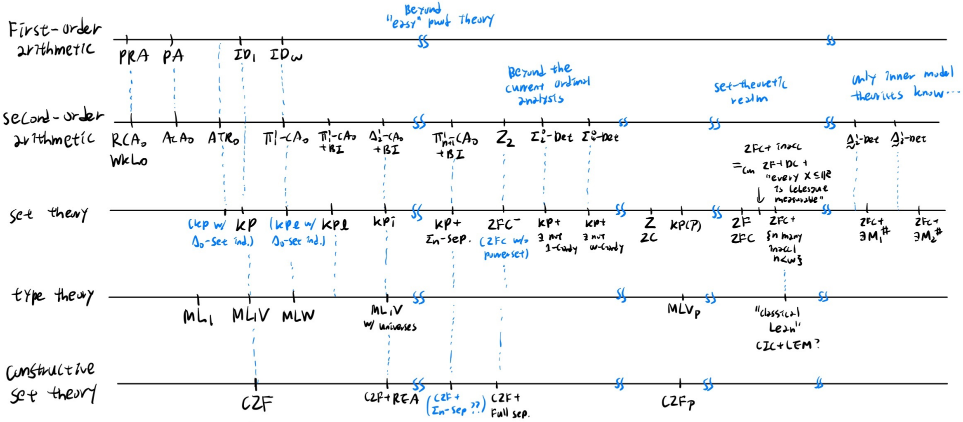
Definition

For two theories S and T extending PRA, define

$$S \leq_{\mathsf{Con}} T \iff \mathsf{PRA} \vdash (\mathsf{Con}(T) \to \mathsf{Con}(S))$$

and

$$S <_{\mathsf{Con}} T \iff T \vdash \mathsf{Con}(S).$$


Beyond "easy" proof theory
First-order arithmetic
PRA
PA
ID1
IDω
Beyond the current ordinal analysis
Set-theoretic realm
Only inner model theorists know...
second-order arithmetic
RCA0 WKL0
ACA0
ATR0
Π¹₁-CA0
Π¹₁-CA0 +BI
Δ¹₂-CA0 +BI
Π¹ₙ₊₁-CA0 +BI
Z2
Σ⁰₂-Det
Σ⁰₄-Det
ZFC+ inacc
=Con ZF+DC + "every X ⊆ ℝ is Lebesgue measurable"
Δ¹₂-Det
Δ¹₃-Det
set theory
(KP w/ Δ0-Set ind.)
KP
(KPℓ w/ Δ0-Set ind.)
KPℓ
KPi
KP+ Σn-Sep.
ZFC⁻ (ZFC w/o powerset)
KP+ ∃ n+1 1-Cardy
KP+ ∃ n+1 ω-Cardy
Z ZC
KP(P)
ZF ZFC
ZFC+ {n many inacc | n<ω}
ZFC+ ∃M1#
ZFC+ ∃M2#
type theory
ML1
ML1V
ML1W
ML1V w/ universes
MLVP
"classical Lean" CIC+LEM?
constructive set theory
CZF
CZF+REA
(CZF + Σn-Sep ??)
CZF+ Full sep.
CZFP

$\leq_{\mathsf{Con}}$ is ill-behaved

Various logicians (Koellner, Simpson, Steel, ...) pointed out that $\leq_{\mathsf{Con}}$ for natural theories is a prewellorder. However,

Theorem (Folklore)

There are theories T_0 and T_1 such that neither $T_0 \leq_{\mathsf{Con}} T_1$ nor $T_1 \leq_{\mathsf{Con}} T_0$.
Also, there are theories $\langle T_n \mid n < \omega \rangle$ such that

$$T_0 >_{\mathsf{Con}} T_1 >_{\mathsf{Con}} T_2 >_{\mathsf{Con}} \cdots .$$

What is wrong?

There are various ways to explain the gap between the facts and the phenomena.
One way is: $\leq_{\mathsf{Con}}$ is too 'finer' than what logicians actually use.

Example

When set theorists prove $S \vdash \mathsf{Con}(T)$, they prove 'S proves T has a transitive model' that is stronger than $S \vdash \mathsf{Con}(T)$.

Proof-theoretic ordinal

Proof-theoretic ordinal gives a linear way to compare theories.
Brief history:

1. (Gentzen 1934) If
$$\varepsilon_0 = \sup\{\omega, \omega^\omega, \omega^{\omega^\omega}, \cdots\}$$
is well-founded, then PA is consistent.
2. (Takeuti 1967) Ordinal analysis of Π^1_1-CA_0.
3. (Arai, Rathjen independently, 1994-1995) Ordinal analysis of Π^1_2-CA_0.
4. (Arai, Pakhomov, Towsner 2024?) Ordinal analysis of the full second-order arithmetic.

Definition

For a theory T, let us define the proof-theoretic ordinal of T by

$$|T|_{\Pi^1_1} = \sup\{\mathrm{otp}(\alpha) : \alpha \text{ is a recursive linear order such that } T \vdash \mathrm{WO}(\alpha)\}.$$

It does not precisely gauge the consistency strength of a theory, e.g., $|T|_{\Pi^1_1} = |T + \mathrm{Con}(T)|_{\Pi^1_1}$.

What does Proof-theoretic ordinal gauge?

The following theorem hints what proof-theoretic ordinal gauges:

Theorem (Kleene, ACA_0)

For every Π^1_1-formula $\phi(X)$ *with all free variables displayed, we can uniformly find a recursive linear order* $\alpha(X)$ *such that*

$$\phi(X) \leftrightarrow \mathsf{WO}(\alpha(X)).$$

i.e., 'Well-foundedness of a recursive linear order' $= \Pi^1_1$.

*A formula of the form "For every real X, (a bounded formula for X)"

Let us define:

1. $T \vdash^{\Sigma^1_1} \phi$ iff $T + \sigma \vdash \phi$ for some true Σ^1_1-sentence σ.
2. $S \subseteq^{\Sigma^1_1}_{\Pi^1_1} T$ iff $S \vdash^{\Sigma^1_1} \phi \implies T \vdash^{\Sigma^1_1} \phi$ for every Π^1_1-sentence ϕ.

Theorem (Walsh 2023)

For Π^1_1-sound theories S, T extending ACA_0,

$$|S|_{\Pi^1_1} \leq |T|_{\Pi^1_1} \iff S \subseteq^{\Sigma^1_1}_{\Pi^1_1} T.$$

That is, comparing proof-theoretic ordinal is equivalent to comparing Π^1_1-consequences of a theory modulo true Σ^1_1-sentences.

Generalizing Proof-theoretic ordinal

Proof-theoretic ordinal gives a linear scale for theories, but its calculation is extremely hard for 'impredicative' theories.
In some sense, Proof-theoretic ordinal as a 'scale' is too 'fine' for impredicative theories.

Observation

For a Π^1_1-sound r.e. theory T extending ACA_0,

$$|T|_{\Pi^1_1} = \sup\{\mathrm{otp}(\alpha) : \alpha \text{ is an arithmetically definable linear order such that } T \vdash \mathrm{WO}(\alpha)\}.$$

Pohlers' δ

We may use T-provably well-founded linear orders over an expansion of $\mathbb{N}$ to gauge the 'performance' of T.

Definition

Let $\mathfrak{M} = (\mathbb{N}; \dots)$ be an expansion of the structure of natural numbers. Suppose that T is an acceptable[†] axiomatization of $\mathfrak{M}$. Define

$$\delta^{\mathfrak{M}}(T) = \sup\{\mathrm{otp}(\alpha) : \alpha \text{ is an } \mathfrak{M}\text{-definable linear order such that } T \vdash \mathrm{WO}(\alpha)\}.$$

[†] T is sound and proves every true atomic sentence and first-order induction scheme over $\mathfrak{M}$.

Spector class

Pohlers focused on the following collection for the expansions:

Definition

A collection Γ of subsets of $\mathbb{N}$ is a Spector class if it satisfies the following:

1. Every atomic predicate and function over $\mathfrak{M}$, and their complements are in Γ. (For functions, consider their graph instead.)
2. Γ contains coding scheme for tuples over $\mathfrak{M}$.
3. Γ is closed under $\cap$, $\cup$, $\exists^0$, $\forall^0$, and trivial combinatorial substitutions.[‡]

‡Trivial combinatorial substitution is a map that is a composition of projection maps and the tuple map.

Definition (continued)

4. Γ has a universal set; That is, for each $n \in \mathbb{N}$ there is an $(n+1)$-ary relation $U \in \Gamma$ such that every n-ary $R \in \Gamma$ is a section of U.
5. Γ has the prewellordering property; That is, for every $P \in \Gamma$ there is a norm $\sigma_P \colon P \to \mathrm{Ord}$ such that the relations
   1. $\vec{m} \le^*_P \vec{n} \iff P(\vec{m}) \land [P(\vec{n}) \to (\sigma(\vec{m}) \le \sigma(\vec{n}))]$, and
   2. $\vec{m} <^*_P \vec{n} \iff P(\vec{m}) \land [P(\vec{n}) \to (\sigma(\vec{m}) < \sigma(\vec{n}))]$

   are both in Γ.

The structures Pohlers considered take the form $(\mathbb{N}; A)_{A\in\Gamma}$ for a Spector class Γ.

Example

Π^1_1 sets and Σ^1_2 sets form Spector classes.

Example

An operator $\mathcal{A}\colon \mathcal{P}(\mathbb{N}) \to \mathcal{P}(\mathbb{N})$ is monotone if $X \subseteq Y \to \mathcal{A}(X) \subseteq \mathcal{A}(Y)$. A least fixed point of F is the $\subseteq$-least set X such that $\mathcal{A}(X) \subseteq X$.
We can construct a least fixed point for a monotone $\mathcal{A}$ as follows:

$$\mathcal{A}^0 = \varnothing,\ \mathcal{A}^\xi = \bigcup_{\eta<\xi} \mathcal{A}(\mathcal{A}^\eta)$$

and $\mathcal{A}^* = \bigcup_{\xi<\omega_1} \mathcal{A}^\xi$. The set of least fixed points of an arithmetical operator forms a Spector class, and it is equal to Π^1_1.

Iterated Spector class

For a collection $\Gamma \subseteq \mathcal{P}(\mathbb{N})$ we can find the the next Spector class

$$\mathsf{SP}(\Gamma) = \bigcap \{\Gamma' \supseteq \Gamma \mid \Gamma' \text{ is a Spector class}\}.$$

Definition

For ξ less than the least recursively inaccessible ordinal, define

1. $\mathsf{SP}^0_{\mathbb{N}} = \varnothing$.
2. $\mathsf{SP}^{\xi+1}_{\mathbb{N}}$ is the next Spector class over $\mathsf{SP}^{\xi}_{\mathbb{N}}$.
3. $\mathsf{SP}^{\delta}_{\mathbb{N}} = \bigcup_{\xi<\delta} \mathsf{SP}^{\xi}_{\mathbb{N}}$ if δ is limit.

$\mathsf{SP}^{\delta}_{\mathbb{N}}$ is not a Spector class when δ is a limit.

Example (Pohlers)

For $\xi \geq 1$ less than the least recursively inaccessible ordinal, we have

$$\delta^{\mathsf{SP}^{\xi}_{\mathbb{N}}}(\mathsf{ACA}_0 + \mathsf{Th}(\mathbb{N}; X)_{X \in \mathsf{SP}^{\xi}_{\mathbb{N}}}) = \varepsilon_{\omega^{\mathsf{CK}}_{\xi}+1}$$

Here $\omega^{\mathsf{CK}}_{\xi}$ is the ξth admissible (or a limit of admissible) ordinal.

Why a dilator?

- Ordinal analysis $\implies$ Analyses Π^1_1-consequences of a theory.
- For a complicated theory, Π^1_n-consequences for $n \geq 2$ affect Π^1_1-consequences.

Why a dilator?

- Ordinal analysis $\implies$ Analyses Π^1_1-consequences of a theory.
- For a complicated theory, Π^1_n-consequences for $n \geq 2$ affect Π^1_1-consequences.
- A dilator is the right concept for Π^1_2-consequences.
- Girard's Π^1_2-proof theory $\implies$ Analyses Π^1_2-consequences of a theory

An example: Class ordinals

Example

There is no transitive class isomorphic with $\mathrm{Ord} + \mathrm{Ord}$, but there is a way to represent it.

Let X be the class of pairs of the form $(0, \xi)$ or $(1, \xi)$ for an ordinal ξ, and impose an order over X as follows:

- $(i, \eta) < (i, \xi)$ iff $\eta < \xi$.
- $(0, \eta) < (1, \xi)$ always holds.

Observation: The above construction is 'uniform.'

Dilators

Let F be a map sending α to the expression for $\alpha + \alpha$. Then

1. We can extend F to a functor from the category of linear orders to the same category.
2. F preserves direct limits and pullbacks.
3. If α is a well-order, then so is $F(\alpha)$.

Dilators

Let F be a map sending α to the expression for $\alpha + \alpha$. Then

1. We can extend F to a functor from the category of linear orders to the same category.
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3. If α is a well-order, then so is $F(\alpha)$.

Definition

A semidilator is a functor from the category of linear orders LO to LO preserving direct limits and pullbacks.
A semidilator F is a dilator if $F(\alpha)$ is a well-order when α is.

Dilators look too 'large,' but it turns out that we can recover a dilator from its small part:

Lemma

Every semidilator is determined by its restriction to the category Nat *of finite ordinals.*

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Definition

A semidilator D is countable if $D(n)$ is countable for each $n \in \mathbb{N}$ (if viewed as objects of the category of finite ordinals.)
A countable semidilator D is A-recursive if we can code the restriction D to Nat into an A-recursive set.

The higher Kleene normal form theorem

Dilators represent Π^1_2-sentences like ordinals represent Π^1_1-sentences.

Theorem (Girard, ACA_0)

For every Π^1_2-formula $\phi(X)$ with all free variables displayed, we can uniformly find a recursive semidilator D_X such that

$$\phi(X) \iff D_X \text{ is a dilator.}$$

Proof-theoretic dilator

Definition

For a theory T, define

$$|T|_{\Pi^1_2} = \sum\{D \mid D \text{ is a recursive semidilator such that } T \vdash D \text{ is a dilator}\}.$$

$|T|_{\Pi^1_2}$ is unique up to bi-embeddability.

Example (Aguilera-Pakhomov)

$|\mathrm{ACA}_0|_{\Pi^1_2} = \varepsilon^+$, where ε^+ is a dilator such that $\varepsilon^+(\alpha)$ is the epsilon number greater than α.

Proof-theoretic dilator and Proof-theoretic ordinal

Theorem (Pakhomov-Walsh, Aguilera-Pakhomov)

Let T be a Π^1_2-sound r.e. theory and α a recursive well-order. Then

$$|T|_{\Pi^1_2}(\alpha) = |T + \mathsf{WO}(\alpha)|_{\Pi^1_1}.$$

Question

What is the proof-theoretic meaning of $|T|_{\Pi^1_2}(\alpha)$ for a non-recursive α?

A comparison

Example (Pohlers)

For $\xi \geq 1$ less than the least recursively inaccessible ordinal, we have

$$\delta^{\mathsf{SP}^{\xi}_{\mathbb{N}}}(\mathsf{ACA}_0 + \mathsf{Th}(\mathbb{N}; \mathsf{SP}^{\xi}_{\mathbb{N}})) = \varepsilon_{\omega^{\mathsf{CK}}_{\xi}+1} = \varepsilon^{+}(\omega^{\mathsf{CK}}_{\xi}).$$

Example (Aguilera-Pakhomov)

$|\mathsf{ACA}_0|_{\Pi^1_2} = \varepsilon^{+}$, where ε^{+} is a dilator such that $\varepsilon^{+}(\alpha)$ is the epsilon number greater than α.

Simplifying Pohlers' framework

$\mathsf{SP}^\xi_\mathbb{N}$ has infinitely many sets, so cumbersome to handle.
We want to find a single set H_ξ so that $(\mathbb{N}, X)_{X\in\mathsf{SP}^\xi_\mathbb{N}}$ and $(\mathbb{N}, H_\xi)$ define the same sets.

Definition

For a real X, the hyperjump of X is the following set

$$\mathsf{HJ}(X) = \{\ulcorner\phi\urcorner \mid \models_{\Pi^1_1} \phi(X) \text{ with all second-order free variables of } \phi \text{ displayed}\},$$

where $\models_{\Pi^1_1}$ is a partial truth predicate for Π^1_1-formulas.

We can see that $\mathsf{SP}^1_{\mathbb{N}} = \Pi^1_1$, $\mathsf{SP}^2_{\mathbb{N}} = \Pi^1_1[\mathsf{HJ}(\emptyset)]$, etc. In general, we have

Theorem

For ξ less than the least recursively inaccessible ordinal, we have $\mathsf{SP}^{\xi+1}_{\mathbb{N}} = \Pi^1_1[\mathsf{HJ}^\xi(\emptyset)]$.

Σ^1_2-singleton real

HJ($\emptyset$) is 'definable' in the following sense:

Definition

A real R is a Σ^1_2-singleton if there is a Σ^1_2-formula $\phi(X)$ such that

$$\phi(R) \land \forall X, Y[\phi(X) \land \phi(Y) \to X = Y].$$

$\mathrm{HJ}^\xi(\emptyset)$ is also a Σ^1_2-singleton for ξ less than the least recursively inaccessible ordinal.

Genedendron

So far, we reduced iterated Spector classes to appropriate Σ^1_2-singletons. We want to introduce a recursive object 'generating'[§] a Σ^1_2-singleton.

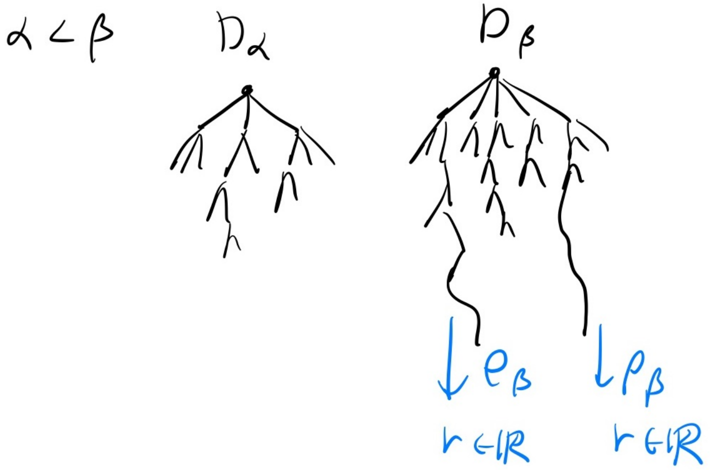
[§] passively or non-deterministically searching?

Definition

A genedendron is a pair (D, ϱ) such that

1. D 'generates' a functorial family $\langle D_\alpha \mid \alpha \in \mathrm{Ord}\rangle$ of trees.
2. ϱ generates a functorial family $\langle \varrho_\alpha \mid \alpha \in \mathrm{Ord}\rangle$, and ϱ_α is a function taking an infinite branch of D_α and returning a real.
3. ϱ_α is a constant function if defined.

We think of D_α a tree and each of the set of immediate successors is linearly ordered. If every set of immediate successors of D_α is well-ordered, we say (D, ϱ) is locally well-founded.



$\alpha < \beta$
D_α
D_β
e_β
ρ_β
$r \in \mathbb{R}$
$r \in \mathbb{R}$

By functoriality, a genedendron is completely determined by $(D_\omega, \varrho_\omega)$.

Definition

A genedendron (D, ϱ) is recursive if there is a recursive set coding $(D_\omega, \varrho_\omega)$.

Example (J.)

We can find a recursive well-founded genedendron (D, ϱ) generating $\mathsf{HJ}(\emptyset)$ such that $D(\alpha)$ is ill-founded iff $\alpha \geq \omega_1^{\mathsf{CK}}$.

$\Pi^1_1[R]$ Proof-theoretic ordinal

Definition

Let R be a Σ^1_2-singleton. For a sound theory T proving 'R uniquely exists,' let us define the $\Pi^1_1[R]$ Proof-theoretic ordinal of T by

$$|T|_{\Pi^1_1[R]} = \sup\{\mathrm{otp}(\alpha) : \alpha \text{ is an } R\text{-recursive linear order such that } T \vdash \mathrm{WO}(\alpha)\}.$$

More precisely, we use the Σ^1_2-singleton definition of R in place of R to formulate the definition.

Pohlers' Characteristic ordinal and $\Pi^1_1[R]$ PTO

Lemma

Let $\xi \geq 1$ be a successor ordinal less than the least recursively inaccessible ordinal. If T is an acceptable axiomatization of $\mathsf{SP}^\xi_{\mathbb{N}}$, then we have

$$\delta^{\mathsf{SP}^\xi_{\mathbb{N}}}(T) = |T|_{\Pi^1_1[\mathsf{HJ}^\xi(\emptyset)]}.$$

The main theorem

Theorem (J.)

Let T be a Π^1_2-sound theory extending ACA_0 and (D, ϱ) be a recursive locally well-founded genedendron generating R.
If T proves (D, ϱ) is a locally well-founded genedendron, and α is an R-recursive well-order such that D_α is ill-founded, then

$$|T|_{\Pi^1_2}(\alpha) = |T[R] + \mathsf{WO}(\alpha)|_{\Pi^1_1[R]}.$$

Here $T[R]$ is the theory T + 'R exists.'

As an instance of the theorem, we can see

$$|\mathsf{ACA}_0 + \text{`}\mathsf{HJ}(\emptyset)\text{ exists'}|_{\Pi^1_1[\mathsf{HJ}(\emptyset)]} = |\mathsf{ACA}_0|_{\Pi^1_2}(\omega_1^{\mathsf{CK}}) = \varepsilon_{\omega_1^{\mathsf{CK}}+1}.$$

Hence we can reproduce Pohlers' result from the proof-theoretic dilator of ACA_0 and appropriate genedendrons.

Also, every ordinal less than

$$\delta^1_2 = \sup\{\mathrm{otp}(\alpha) \mid \alpha \text{ is a } \Delta^1_2\text{-wellorder}\} = \min\{\sigma \mid L_\sigma \prec_{\Sigma_1} L\}$$

is isomorphic to an R-recursive well-order for some Σ^1_2-singleton real R.
Hence the previous theorem provides the proof-theoretic meaning of $|T|_{\Pi^1_2}(\alpha)$ for $\alpha < \delta^1_2$.

Question: Σ^1_2-altitude

We can define the 'ordinal complexity' for Σ^1_2-singletons:

Definition

For a Σ^1_2-singleton R, let us define

$$\mathsf{Alt}_{\Sigma^1_2}(R) = \min\{\alpha \mid \exists(D,\varrho)[(D,\varrho) \text{ is a genedendron generating } R \text{ and } D_\alpha \text{ illfounded.}]\}$$

Question

$$\mathsf{Alt}_{\Sigma^1_2}(R) = \min\{M \cap \mathrm{Ord} \mid M \vDash \mathsf{ATR}_0^{\mathrm{set}} \land R \text{ is } \Sigma_1\text{-definable over } M\}$$

Any other Questions?

(Removed due to copyright issues)

Thank you!