Pre-independence relations induced by Morley sequences in $NSOP_1$ theories

Joonhee Kim
joint work with Byunghan Kim and Hyoyoon Lee

Korea Institute for Advanced Study

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In a given mathematical structure $\mathbb{M}$ and a language $\mathcal{L}$, an indiscernible sequence is a sequence $(a_i)_{i<\omega}$ in $\mathbb{M}$ that has some sort of "consistent tendency" with respect to $\mathcal{L}$. Precisely, we say a sequence $(a_i)_{i<\omega}$ is indiscernible over a set A if

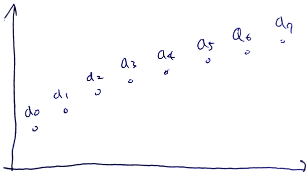
$$a_{i_0} \dots a_{i_{n-1}} \equiv_A^{\mathcal{L}} a_{j_0} \dots a_{j_{n-1}}$$

for all $i_0 < \cdots < i_{n-1}$ and $j_0 < \cdots < j_{n-1}$ in ω. It means that those two finite sequences satisfy exactly the same formulas in $\mathcal{L}(A)$.

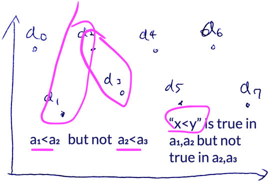
Example

Consider $(\mathbb{R}, <)$.

- Monotonically increasing/decreasing sequences are indiscernible over $\emptyset$.
- If a sequence is oscillating, then in is not indiscernible.



indiscernible in (R,<)



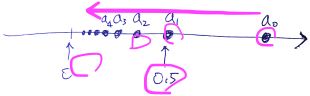
not indiscernible in (R,<)

The base set of a given indiscernible sequence (A in the above definition) can be regarded as an "observer". The same sequence may or may not be an indiscernible sequence, depending on how we choose the base set.

Example

In $(\mathbb{R}, <)$, let $(a_n) = 1/n$ for each $n < \omega$. Then $(a_n)_{n<\omega}$ is indiscernible over $\{0\}$ but not indiscernible over $\{0.5\}$.

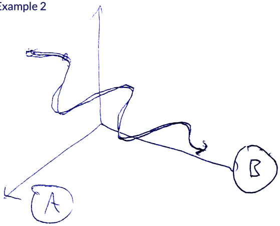
Example 1

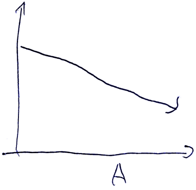


$a_0 \not\equiv_{\{0.5\}} a_2$

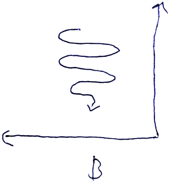
$a_0 \models 0.5 < x$ but $a_2 \not\models 0.5 < x$

Example 2



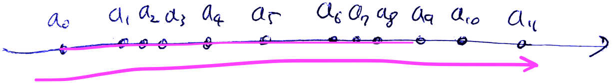


indiscernible over A



not indiscernible over B

If the language $\mathcal{L}$ becomes richer and can express a wider variety of movements of a sequence, then the sequences become harder to be indiscernible.

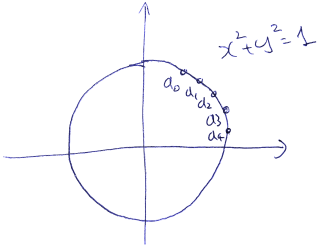


indiscernible in (R,<) but not indiscernible in (R,<,d(x,y))
where d(x,y) is the distance between x and y

Fact

If an equation (formula) $\varphi(x_0, \dots, x_{n-1})$ has infinitely many solutions, then there exists an indiscernible sequence $(\bar{a}_i)_{i<\omega}$ such that $\models \varphi(\bar{a}_i)$ for all $i < \omega$.

For any given mathematical object defined by the language, one can consider indiscernible sequences living in the object if it has infinitely many elements.



Fact

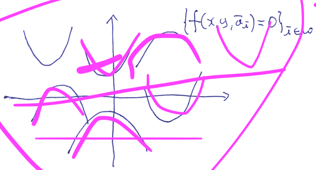
If $a \in A$, then every indiscernible sequence $(a_i)_{i<\omega}$ over A with $a_0 = a$ is constant (i.e., $a_i = a_j$ for all $i, j < \omega$). Since $x = a \in \mathcal{L}(A)$, if such an indiscernible sequence $(a_i)_{i<\omega}$ exists, then $a_i \models x = a$ for all $i < \omega$.

Example

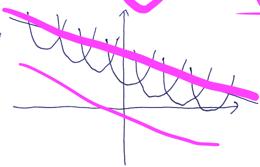
Let C be a curve on a plane and consider all straight lines that intersect to C at two points. If C can be defined by an equation $f(x, y, \bar{a}) = 0$, then the set of all such straight lines can be defined by

$$\varphi(x, y, \bar{a}) = \exists x_0, x_1, y_0, y_1 \Big((x_0 \neq x_1 \vee y_0 \neq y_1) \wedge \bigwedge_{i<2} f(x_i, y_i, \bar{a}) = 0 \wedge \bigwedge_{i<2} y_i = x x_i + y \Big)$$

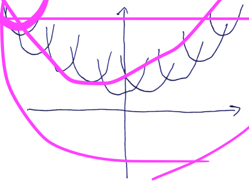
The set of straight lines depends on the choice of $\bar{a}$, the coefficients of the equation f. If a sequence $(\bar{a}_i)_{i<\omega}$ has a consistent tendency (is indiscernible), then the definable sets $\varphi(x, y, \bar{a}_0), \varphi(x, y, \bar{a}_1), \ldots$ are more likely to have an intersection.



Case 1. $(\bar{a}_i)_{i<\omega}$ is not indiscernible and $\{\varphi(x, y, \bar{a}_i)\}_{i<\omega}$ has no common solution.



Case 2. $(\bar{a}_i)_{i<\omega}$ is indiscernible and $\{\varphi(x, y, \bar{a}_i)\}_{i<\omega}$ has common solutions.



Case 3. $(\bar{a}_i)_{i<\omega}$ is indiscernible but $\{\varphi(x, y, \bar{a}_i)\}_{i<\omega}$ has no common solution.

Definition

We say a formula $\varphi(\bar{x}, \bar{a})$ divides over a set A if there exists an indiscernible sequence $(\bar{a}_i)_{i<\omega}$ over A with $\bar{a}_0 = \bar{a}$ such that $\{\varphi(\bar{x}, \bar{a}_i) : i < \omega\}$ has no common solution.

By using this we can define pre-independence relation (invariant ternary relation) $\mathop{\smile\!\!\!\!\!\!\mid}^d$ as follows.

Definition [Non-dividing independence]

We write $a \mathop{\smile\!\!\!\!\!\!\mid}^d_C b$ if there is no dividing formula $\varphi(x) \in \mathcal{L}(Cb)$ over C such that $a \models \varphi(x)$. We define $\mathop{\smile\!\!\!\!\!\!\mid}^f$ (non-forking independence) as the weakest pre-independence relation stronger than $\mathop{\smile\!\!\!\!\!\!\mid}^d$ (i.e. $\mathop{\smile\!\!\!\!\!\!\mid}^f \rightarrow \mathop{\smile\!\!\!\!\!\!\mid}^d$) satisfying right extension.

If we fix a base set A, then the non-dividing independence $\mathop{\smile\!\!\!\!\!\!\mid}^d_A$ can be regarded as a binary relation (over A).

Fact

In algebraically closed fields $K \subseteq L$ and $a, b \in L$, $a \mathop{\smile\!\!\!\!\!\!\mid}^d_K b$ if and only if a and b are algebraically independence over K. Moreover, if a sequence $(a_i)_{i<\omega}$ satisfies $a_i \mathop{\smile\!\!\!\!\!\!\mid}^d_K a_{<i}$ for all $i < \omega$, then $(a_i)_{i<\omega}$ are algebraically independence over K, and vice versa. It is known that $\mathop{\smile\!\!\!\!\!\!\mid}^d = \mathop{\smile\!\!\!\!\!\!\mid}^f$ in ACF.

Idea

$\varphi(x, b)$ divides over C

$\Rightarrow$ $\{\varphi(x, b_i)\}_{i<\omega}$ has no common solution for some indiscernible sequence $(b_i)_{i<\omega}$ over C with $b_0 = b$.

$\Rightarrow$ If $\{\varphi(x, b_i)\}_{i<\omega}$ has no common solution even though $(b_i)_{i<\omega}$ is indiscernible (moving with consistency tendency), then we may consider $\varphi(x, b)$ to be 'small', or to satisfy some property that can be metaphorically called 'smallness'.

Idea

$a \downarrow^d b$

$\Rightarrow$ There is no dividing ('small') formula $\varphi(x) \in \mathcal{L}(Cb)$ over C capturing a.

$\Rightarrow$ a is 'relatively free' from b over C.

$\Rightarrow$ We may consider a to be 'independent' from b over C.

Fact

In ACF, $a \mathop{\smile\!\!\!\!\!\mid}^f_C b$ if and only if a and b are algebraically independence over C.

In model theory, there is a class of mathematical structures (theories) called stable, which are, roughly speaking, generalizations of the features of ACF.

Fact

In stable theories, $\mathop{\smile\!\!\!\!\!\mid}^f$ satisfies the following.

- Monotonicity: If $aa' \mathop{\smile\!\!\!\!\!\mid}^f_C bb'$, then $a \mathop{\smile\!\!\!\!\!\mid}^f_C b$.
- Base monotonicity: If $a \mathop{\smile\!\!\!\!\!\mid}^f_C bb'$, then $a \mathop{\smile\!\!\!\!\!\mid}^f_{Cb} b'$.
- Transitivity: If $a \mathop{\smile\!\!\!\!\!\mid}^f_{Db} c$ and $b \mathop{\smile\!\!\!\!\!\mid}^f_D c$, then $ab \mathop{\smile\!\!\!\!\!\mid}^f_D c$.
- Right extension: If $a \mathop{\smile\!\!\!\!\!\mid}^f_D b$, then for all c, there exists $c' \equiv_{Db} c$ such that $a \mathop{\smile\!\!\!\!\!\mid}^f_D bc'$.
- Existence: $a \mathop{\smile\!\!\!\!\!\mid}^f_C \emptyset$ for all $a \notin \operatorname{acl}(C)$
- Symmetry: If $a \mathop{\smile\!\!\!\!\!\mid}^f_C b$, then $b \mathop{\smile\!\!\!\!\!\mid}^f_C a$.
- Uniqueness: If $a \mathop{\smile\!\!\!\!\!\mid}^f_M B$, $a' \mathop{\smile\!\!\!\!\!\mid}^f_M B$, and $a \equiv_M a'$, then $a \equiv_{MB} a'$.
- Strong finite character: If $a \not\mathop{\smile\!\!\!\!\!\mid}^f_C b$, then there is $\varphi(x, y)$ such that $\varphi(x, b) \in \operatorname{tp}(a/Cb)$ and $a' \not\mathop{\smile\!\!\!\!\!\mid}^f_C b$ for all $a' \models \varphi(x, b)$.
- Independence theorem: If $a \mathop{\smile\!\!\!\!\!\mid}^f_C b$, $a' \mathop{\smile\!\!\!\!\!\mid}^f_C b'$, $b \mathop{\smile\!\!\!\!\!\mid}^f_C b'$, and $a \equiv^L_C a'$, then there is a'' such that $a'' \equiv^L_{Cb} b$, $a'' \equiv^L_{Cb'} a'$, and $a'' \mathop{\smile\!\!\!\!\!\mid}^f_C bb'$.

A class of simple theories is a larger class than the class of stable theories. Roughly speaking, simplicity can be thought of as stability plus randomness.

Fact [Kim, Pillay, 1997]

In simple theories, $\mathop{\smile\!\!\!\!\!\!\;|}^{f}$ satisfies monotonicity, right extension, strong finite character, base monotonicity, left transitivity, existence, symmetry over sets, and the independence theorem over models.

In a study of NSOP$_1$ theories, a bigger class than the class of simple theories, Kaplan and Ramsey introduced $\mathop{\smile\!\!\!\!\!\!\;|}^{K^{\perp^i}}$ and proved the following.

Fact [Kaplan, Ramsey, 2017]

In NSOP$_1$ theories, $\mathop{\smile\!\!\!\!\!\!\;|}^{K^{\perp^i}}$ satisfies monotonicity, right extension, strong finite characte, existence, symmetry, and the independence theorem over models.

Fact [Kruckman, Ramsey, 2023] [Hanson 2023]

In any theory,

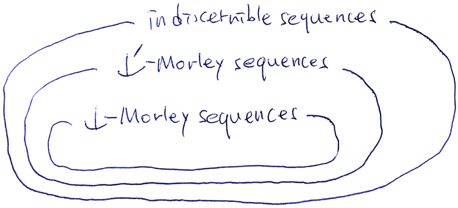
- $\mathop{\smile\!\!\!\!\!\!\;|}^{f}$ is stronger than $\mathop{\smile\!\!\!\!\!\!\;|}^{K^{\perp^i}}$,
- $\mathop{\smile\!\!\!\!\!\!\;|}^{f}$ satisfies monotonicity, right extension, strong finite character, base monotonicity, and left transitivity over sets,
- $\mathop{\smile\!\!\!\!\!\!\;|}^{K^{\perp^i}}$ satisfies monotonicity, right extension, strong finite character, and existence over sets.

Definition

Let $\mathop{\smile\!\!\!\!\!\!\mid}$ be a pre-independence relation. A sequence $(a_i)_{i<\omega}$ is $\mathop{\smile\!\!\!\!\!\!\mid}$-Morley sequence over B if

- it is indiscernible over B,
- $a_i \mathop{\smile\!\!\!\!\!\!\mid}_B a_{<i}$ for all $i < \omega$.

The class of $\mathop{\smile\!\!\!\!\!\!\mid}$-Morley sequences is a subclass of the class of indiscernible sequences. If $\mathop{\smile\!\!\!\!\!\!\mid}$ is stronger than $\mathop{\smile\!\!\!\!\!\!\mid}'$, then the class of $\mathop{\smile\!\!\!\!\!\!\mid}$-Morley sequences is a subclass of the class of $\mathop{\smile\!\!\!\!\!\!\mid}'$-Morley sequences.



$\mathop{\smile\!\!\!\!\!\!\mid} \to \mathop{\smile\!\!\!\!\!\!\mid}'$

Definition

A formula $\varphi(x,a)$ $\mathop{\smile\!\!\!\!\!\!\!\perp}$-Kim-divides over B if there exists a $\perp$-Morley sequence $(a_i)_{i<\omega}$ over B with $a_0 = a$ such that $\{\varphi(x,a_i)\}_{i<\omega}$ has no common solution.

As it is harder to be a $\perp$-Morley sequence than be an indiscernible sequence, it is harder to $\perp$-Kim-divide than divide. So if $\varphi(x,a)$ $\perp$-Kim-divides, then we may consider it to be 'smaller' than dividing formulas.

Definition [Non-$\perp$-Kim-dividing independence]

We write $a \perp_C^{Kd^{\perp}} b$ if there is no $\perp$-Kim-dividing formula $\varphi(x) \in \mathcal{L}(Cb)$ such that $a \models \varphi(x)$. We define $\perp^{K^{\perp}}$ as the weakest pre-independence relation stronger than $\perp^{Kd^{\perp}}$ (i.e. $\perp^{K^{\perp}} \to \perp^{Kd^{\perp}}$) satisfying right extension.

In a similar argument to what we discussed about $\perp^d$ above, $a \perp_C^{Kd^{\perp}} b$ means that a is not captured by $\perp$-Kim-dividing formula in $\mathcal{L}(Cb)$ over C, hence we may consider a to be independent from b over C. But this independence is weaker than $\perp^d$-independence since $\perp$-Kim-dividing formulas are smaller than dividing formulas.

Fact

- $\perp^d \to \perp^{Kd^{\perp}}$ and $\perp^f \to \perp^{K^{\perp}}$ for all pre-independence relation $\perp$.
- $\perp^{K^{\perp}} \to \perp^{K^{\perp'}}$ for all $\perp' \to \perp$.

Question

Is there a pre-independence relation $\mathop{\smile\!\!\!\!\!\!|}$ such that

- $\mathop{\smile\!\!\!\!\!\!|}^{f} \to \mathop{\smile\!\!\!\!\!\!|} \to \mathop{\smile\!\!\!\!\!\!|}^{K^{\mathop{\smile\!\!\!\!\!\!|}^{i}}}$,
- $\mathop{\smile\!\!\!\!\!\!|} = \mathop{\smile\!\!\!\!\!\!|}^{f}$ over sets in simple theories,
- $\mathop{\smile\!\!\!\!\!\!|} = \mathop{\smile\!\!\!\!\!\!|}^{K^{\mathop{\smile\!\!\!\!\!\!|}^{i}}}$ over models in NSOP_1 theories,
- $\mathop{\smile\!\!\!\!\!\!|}$ satisfies monotonicity, right extension, strong finite character, existence, symmetry over sets, and the independence theorem over models in NSOP_1 theories?

Fact

- $\mathop{\smile\!\!\!\!\!\!|}^{f} \to \mathop{\smile\!\!\!\!\!\!|}^{K^{\mathop{\smile\!\!\!\!\!\!|}^{f}}} \to \mathop{\smile\!\!\!\!\!\!|}^{K^{\mathop{\smile\!\!\!\!\!\!|}^{i}}}$,
- $\mathop{\smile\!\!\!\!\!\!|}^{K^{\mathop{\smile\!\!\!\!\!\!|}^{f}}} = \mathop{\smile\!\!\!\!\!\!|}^{f}$ over sets in simple theories,
- $\mathop{\smile\!\!\!\!\!\!|}^{K^{\mathop{\smile\!\!\!\!\!\!|}^{f}}} = \mathop{\smile\!\!\!\!\!\!|}^{K^{\mathop{\smile\!\!\!\!\!\!|}^{i}}}$ over models in NSOP_1 theories.
- $\mathop{\smile\!\!\!\!\!\!|}^{K^{\mathop{\smile\!\!\!\!\!\!|}^{f}}}$ satisfies monotonicity, right extension, strong finite character over sets in any theory.

Theorem [Kim, K, Lee]

$\mathop{\smile\!\!\!\!\!\!|}^{K^{\mathop{\smile\!\!\!\!\!\!|}^{f}}}$ satisfies existence over sets in NSOP_1 theories. There exists a mathematical structure such that $\mathop{\smile\!\!\!\!\!\!|}^{K^{\mathop{\smile\!\!\!\!\!\!|}^{f}}}$ does not satisfies existence over sets.

Fact [Dobrowolski, Kim, Ramsey, 2020]

In NSOP_1 theories, if we assume that $\mathop{\smile\!\!\!\!\!\mid}^f$ satisfies existence over sets, then $\mathop{\smile\!\!\!\!\!\mid}^{K^{\mathop{\smile\!\!\!\!\!\mid}^f}}$ satisfies symmetry over sets and independence theorem over models.

Question

- Does $\mathop{\smile\!\!\!\!\!\mid}^f$ satisfy existence over sets in NSOP_1 theories?
- Can we show that $\mathop{\smile\!\!\!\!\!\mid}^{K^{\mathop{\smile\!\!\!\!\!\mid}^f}}$ satisfies symmetry and the independence theorem without assuming existence of $\mathop{\smile\!\!\!\!\!\mid}^f$?

References

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