An introduction to model companion : Infinite sets are not trivial!

Junguk Lee

Changwon National University

The 3rd Korea Logic Day
January 12th 2024

- My goal is to convince that infinite sets are not trivial (in a model theoretic view point).
- I will introduce a notion of **model companion** from model theory.
- In the term of model companion, infinite sets are special as like
  - dense linear order without end points between linear orders,
  - random graphs between graphs,
  - algebraic closed fields between fields.

Some conventions

- Fix a language $\mathcal{L}$.
- Given a structure $\mathcal{M} = (M; \ldots)$, we use the underlying set M for the structure $\mathcal{M}$.
- We use $x, y, z, \ldots$ for a tuple of variables if there is no confusion.
- We mean $a \in M$ a tuple of elements in M^1.
- Given a formula $\varphi(x)$ and $a \in M^{|x|}$, we write

$$\mathcal{M} \models \varphi(a)$$

if $\varphi(a)$ holds in $\mathcal{M}$. Conventionally, let us call such a tuple a solution of $\varphi(x)$.

What to do?

- Fix a theory T.
- Let
$$(\mathcal{C} =)\mathcal{C}_T := \{M : M \models T\}$$
be the class of all models of T.
- I will take a specific class $\mathcal{C}^*$ related to $\mathcal{C}$, which is **elementary**.
- That is, there is a theory T^* such that
$$\mathcal{C}^* = \{M^* \in \mathcal{C} : M^* \models T^*\}.$$
- I try to convince that the taken class $\mathcal{C}^*$ is *special* in some sense.

- Let $\mathcal{L} = \emptyset$ and let $T = \emptyset$.
- Then, $\mathcal{C}$ is the class of all sets without any structures except the equality.
- Let $\mathcal{C}^*$ be the class of all infinite sets, which is **elementary**.
- Then, $\mathcal{C}^*$ is special in the following way:
  - $(\forall M \in \mathcal{C})(\exists M^* \in \mathcal{C}^*)(M \hookrightarrow M^*)$.
  - $(\forall M^* \in \mathcal{C}')(\exists M \in \mathcal{C})(M^* \hookrightarrow M)$.
  - Any structure in $\mathcal{C}^*$ is existentially closed in the following sense:

- Take $M^* \in \mathcal{C}^*$, which is just a infinite set arbitrary.
- Let $\varphi(x, y)$ be a **quantifier-free formula**. That is, $\varphi(x, y)$ says about (in)equalities between variables in the tuples x and y.
- Consider the case $x = (x_1)$ and $y = (y_1, y_2)$.
- Take $b \in (M^*)^{|y|}$ arbitrary.
- Suppose there is an extension $M \in \mathcal{C}$ of M^* such that

$$M \models \exists x(\varphi(x, b)).$$

  Let $a \in M$ be a solution of $\varphi(x, b)$.
- WLOG, by the disjunctive normal form (DNF), we may assume that $\varphi(x, y)$ is equivalent to one of the following:

$$\begin{cases} x = y_1 \wedge x = y_2, \\ x = y_1 \wedge x \neq y_2, \\ x \neq y_1 \wedge x = y_2, \\ x \neq y_1 \wedge x \neq y_2. \end{cases}$$

- For example, $\varphi(x, y) \equiv x \neq y_1 \wedge x \neq y_2$ and

$$M \models a \neq b_1 \wedge a \neq b_2.$$

- Then, since M^* is infinite, we can find a^* in M^* such that $a^* \neq b_1, b_2$, which is a solution of $\varphi(x, b)$, and so

$$M^* \models \exists x(\varphi(x, b)).$$

- In summary, if a system of (in)equations over M^* is consistent in $\mathcal{C}$, then it has already a solution in M^*.

- Let $\mathcal{L} = \{<\}$ and let T be the theory of linear orders.
- Then, $\mathcal{C}$ is the class of all linear orders.
- Let $\mathcal{C}^*$ be the class of all dense linear orders without endpoints, which is elementary.
- Denote DLO by the theory of dense linear orders without endpoints.
- Then, $\mathcal{C}^*$ is special as like infinite sets, that is,
  - $(\forall M \in \mathcal{C})(\exists M^* \in \mathcal{C}^*)(M \hookrightarrow M^*)$.
  - $(\forall M^* \in \mathcal{C}^*)(\exists M \in \mathcal{C})(M^* \hookrightarrow M)$.
  - Any linear order in $\mathcal{C}^*$ is existentially closed in $\mathcal{C}$.

- Take $M^* \models DLO$.
- Let $\varphi(x, y)$ be a quantifier-free formula. That is, $\varphi(x, y)$ is a system of inequalities between variables x and y.
- Consider the case $x = (x_1)$ and $y = (y_1, y_2)$.
- Take $b \in (M^*)^{|y|}$ arbitrary.
- Suppose there is an extension $M \in \mathcal{C}$ of M^* such that

$$M \models \exists x(\varphi(x, b)).$$

  Let $a \in M$ be a solution of $\varphi(x, b)$.
- WLOG, we may assume that for $b = (b_1, b_2)$, $b_1 < b_2$ and $a \neq b_1, b_2$. By DNF, we may assume that $\varphi(x, y)$ is equivalent to one of the following:

$$\begin{cases} x < y_1 < y_2, \\ y_1 < x < y_2, \\ y_1 < y_2 < x. \end{cases}$$

- Since M^* is dense and has no end points, we can find a solution of $\varphi(x, b)$, that is,
$$M^* \models \exists x(\varphi(x, b)).$$

- Let $\mathcal{L}_{ring} = \{+, \cdot, 0, 1\}$ be the ring language and T be the theory of field.
- Let $\mathcal{C}$ be the class of fields satisfying

$$\exists x \forall y (x \neq y^2).$$

- Let $\mathcal{C}^*$ be the class of algebraically closed fields, which is elementary.
- Then, $\mathcal{C}^*$ is special as like infinite sets and DLO.
    - $(\forall M \in \mathcal{C})(\exists M^* \in \mathcal{C}^*)(M \hookrightarrow M')$.
    - $(\forall M^* \in \mathcal{C}^*)(\exists M \in \mathcal{C})(M^* \hookrightarrow M)$.
    - Any field in $\mathcal{C}^*$ is existentially closed in $\mathcal{C}$.
- Indeed,
    - Given a field $K \in \mathcal{C}$, the algebraical closure $K^* \in \mathcal{C}^*$.
    - Given $K^* \in \mathcal{C}^*$, $K^*(t) \in \mathcal{C}$ for a transcendental element t outside K^*.

- Let M^* be an algebraically closed field.
- Let $\varphi(x, y)$ be a quantifier-free formula.
- WLOG, by DNF, we may assume that

$$\varphi(x, y) \equiv \bigvee \varphi_i(x, y)$$

where each $\varphi_i(x, y)$ is of the form: For some $f_1(y; x), \ldots, f_n(y; x)$ and $g(y; x)$ in $\mathbb{Z}[y][x]$,

$$\bigwedge f_k(y; x) = 0 \wedge g(y; x) \neq 0,$$

that is, a system of (in)equations of polynomials in $\mathbb{Z}[y][x]$.
- So, each $\varphi_i(x, y)(\equiv \bigwedge f_k(y; x) = 0 \wedge g(y; x) \neq 0)$ parametrizes Zariski open subsets of the algebraic set defined by the system of equations

$$\bigwedge f_k(y; x) = 0$$

in the parameter y.

- Take $b \in (M^*)^{|y|}$ such that $\varphi(x, b)$ is consistent.
- That is, there is a field M extending M^* such that $\varphi(M, b) \neq \emptyset$ and so one of Zariski open sets $\varphi_i(M, b)$ is non-empty.
- Note that for any system S of (in)equations of polynomials over M^*, the M^*-rational points of S is Zariski dense.
- So, for each i, $\varphi_i(M^*, b)$ is a Zariski dense subset of $\varphi_i(M, b)$.
- Thus, $\varphi(M^*, b) \neq \emptyset$ because one of Zariski open sets $\varphi_i(M, b)$ is non-empty, and so

$$M^* \models \exists x \, (\varphi(x, b)) \, .$$

- In '56, Robinson introduced a notion of **model companion**, which isolate a common phenomena seen in infinite sets, DLOs, and algebraically closed fields.
- Let T be a theory in $\mathcal{L}$.
- A $\mathcal{L}$-theory T^* is called **model companion** if
  - $(\forall M \models T)(\exists M^* \models T^*)(M \hookrightarrow M^*)$.
  - $(\forall M^* \models T^*)(\exists M \models T)(M' \hookrightarrow M)$.
  - Each $M^* \models T^*$ is existentially closed.
- And such a model-companion T^* of T is unique if it exists.
- Note that it is not necessary that T^* extends T.
- For example, consider the following:
  - T is the theory of fields satisfying

$$\exists x \forall y (x \neq y^2).$$

  - T^* is the theory of algebraically closed field.

Syntactical results on the model companion

- Let T^* be the model companion of T (if exists).
- We have the following syntactical results on the model companion.
- First, T^* is axiomatized by $\forall\exists$-sentences.
- Second, any formula is equivalent to a universal formula modulo T^*. That is, given a formula $\varphi(x)$, there is a quantifier free formula $\psi(x, y)$ such that

$$T^* \models \varphi(x) \leftrightarrow \forall y \psi(x, y).$$

Example

1. In $\mathcal{L} = \{E\}$, the theory of the random graph is the model companion of the theory of graphs.
2. In the language $\mathcal{L}_{or} = \mathcal{L}_{ring} \cup \{<\}$ of ordered rings, *RCF* is the model companion of the theory of ordered fields.
3. In the language $\mathcal{L}_{dr} = \mathcal{L}_{ring} \cup \{\partial\}$ of differential field, *DCF* is the model companion of the theory of differential fields.
4. In the language $\mathcal{L}_{gp} = \{\cdot, ^{-1}, 1\}$ of group, the theory of divisible abelian groups is the model companion of the theory of torsion-free abelian groups.

- In $\mathcal{L}_{gp}$, there is NO model-companion of the theory of groups.
- Indeed, there is an existentially closed group but being existentially closed between all groups is NOT elementary.
- We first review several consequences of being existentially closed.
- Let G be a existentially closed group.
- First, we show that for any $a \in G$,

$$\mathrm{Aut}(G)a = \mathrm{Inn}(G)a,$$

  where $\mathrm{Inn}(G) := \{g : x \mapsto g^{-1}xg : g \in G\}$.
- Namely, let $a, b \in G$ and $\sigma \in \mathrm{Aut}(G)$ with $b = \sigma(a)$.
- Consider the group $\mathrm{Aut}(G) \ltimes G$ with $\sigma(g) = \sigma^{-1}g\sigma$ for $\sigma \in \mathrm{Aut}(G)$ and $g \in G$.
- Consider the existential G-formula

$$\exists x(x^{-1}ax = b).$$

- Then, $\mathrm{Aut}(G) \ltimes G \models \exists x(x^{-1}ax = b)$ and $G \subset \mathrm{Aut}(G) \ltimes G$. Thus,

$$G \models \exists x(x^{-1}ax = b).$$

- Second, we show that for any cyclic group C_p of prime order p,

$$C_p < G$$

- Consider the existential formula

$$\exists x(x \neq e \wedge x^p = e).$$

- Then, $C_p \times G \models \exists x(x \neq e \wedge x^p = e)$. Thus,

$$G \models \exists x(x \neq e \wedge x^p = e).$$

- Now, we assume that there is a model companion T^* of the theory of groups.
- Let $G \models T^*$ be κ-saturated and strongly κ-homogeneous for some $\kappa > (2^{\aleph_0})^+$, and so it is existentially closed.
- Consider the equivalence relation $x \equiv y$, which is definable by the formula $\exists z(x = zyz^{-1})$.
- Since $\mathcal{L}_{gp}$ is countable,

$$|G/\equiv| \leq 2^{\aleph_0}.$$

- Thus, $G/\equiv$ is finite. Namely, if $G/\equiv$ is infinite, by compactness, $|G/\operatorname{Aut}(G)| \geq \kappa$, which is impossible because $|G/\equiv| \leq 2^{\aleph_0} < \kappa$.
- But $G/\equiv$ is infinite because for each prime p, $C_p \subset G$ and for primes $p \neq p'$, $C_p \ncong C_{p'}$.

[1] A. Robinson,
Complete theories, North-Holland Publishing Co., Amsterdam, (1956), 129 pp.

[2] A. Robinson,
On the concept of a differentially closed field, *Bull. Res. Council Israel Sect.*, (1959), 113-128.

[3] K. Tent and M. Ziegler,
A course in model theory, Lecture Notes in Logic, **40**, Cambridge University Press, 2012, 248 pp.