Word problem for groups and G-automata

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Introduction

In group theory, a group is often given by a finite presentation, such as

$$G = \langle\, x_1, \ldots, x_n \mid r_1, \ldots, r_m \,\rangle.$$

Examples

- $\mathbb{Z}^2 = \langle\, x, y \mid xyx^{-1}y^{-1} \,\rangle$ (the free abelian group of rank 2),
- $S_3 = \langle\, x, y \mid x^2, y^2, (xy)^3 \,\rangle$ (the symmetric group of degree 3, order 6).

For a given finite presentation of a group H, the word problem of H (w.r.t. the presentation) is the following decision problem:

Word problem of H (Dehn, 1911)

Input: two words u, v on the generators

Question: does $u = v$ in H? ($\iff u^{-1}v = 1_H$)

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3. Simplified proof of EKO's theorem

Word problem for groups (1/2)

Definition (word problem)

Let H be a finitely generated group, Σ be a finite alphabet, and $\rho\colon \Sigma^* \to H$ be a **surjective** monoid homomorphism. The word problem of H is defined as $\mathrm{WP}_\rho(H) := \rho^{-1}(1_H)$, where $1_H \in H$ is the identity element.

Although the word problem $\mathrm{WP}_\rho(H)$ depends on the choice of ρ, we can ignore ρ in most cases:

Remark

Let $\mathcal{C}$ be a class of languages being closed under inverse homomorphism. If $\mathrm{WP}_\rho(H) \in \mathcal{C}$ for **some** $\rho\colon \Sigma^* \to H$, then so is for **every** ρ.

So we often write just $\mathrm{WP}(H)$.

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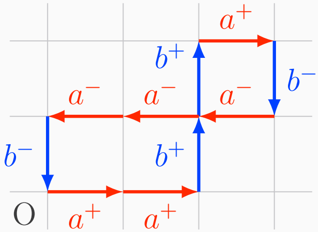
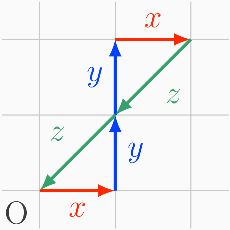
Word problem for groups (2/2)

Examples

Let $H = \mathbb{Z}^2$.

- Let $\Sigma = \{a^+, a^-, b^+, b^-\}, \rho(a^\pm) = (\pm 1, 0), \rho(b^\pm) = (0, \pm 1)$. Then
  $\mathrm{WP}_\rho(H) = \{ w \in \Sigma^* \mid w$ has same # of a^+ (resp. b^+) and a^- (resp. b^-) $\}$
  $\ni a^+a^+b^+b^+a^+b^-a^-a^-a^-b^-$ (below left)
- Let $\Sigma = \{x, y, z\}, \rho(x) = (1, 0), \rho(y) = (0, 1), \rho(z) = (-1, -1)$. Then
  $\mathrm{WP}_\rho(H) = \{ w \in \Sigma^* \mid w$ has same # of $x, y, z \}$
  $\ni xyyxzz$ (below right)





Known results

Theorem (Anīsīmov, 1972)

For a finitely generated group H, the following are equivalent:

1. The word problem $\mathrm{WP}(H)$ of H is a regular language.
2. H is a finite group.

Theorem (Muller–Schupp (1983) $+$ Dunwoody (1985))

For a finitely generated group H, the following are equivalent:

1. The word problem $\mathrm{WP}(H)$ of H is a context-free language.
2. H is a virtually free group (i.e., has a finite index free subgroup).

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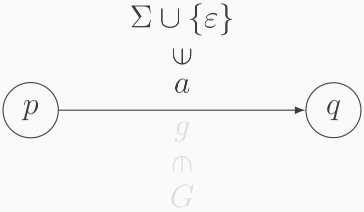
G-automata (1/3) — Informal definition

Let G be a group. Roughly speaking,

$$G\text{-automaton} = \text{usual NFA} + (\text{edge labels} \in G).$$

For each edge e of a G-automaton:



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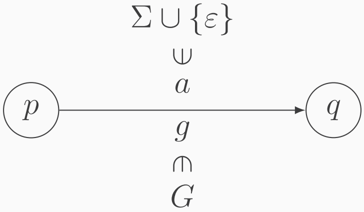

G-automata (2/3) — Convention for graphs and paths

- A graph is a 4-tuple $\Gamma = (V, E, \mathsf{s}, \mathsf{t})$, where
  - V is the vertex set,
  - E is the edge set,
  - $\mathsf{s}\colon E \to V$ is a function assigning to every edge its source, and
  - $\mathtt{t}\colon E \to V$ is a function assigning to every edge its target.
- A path in a graph Γ is a **sequence** $e_1 \cdots e_n \in E^*$ **of edges** such that $\mathsf{t}(e_i) = \mathsf{s}(e_{i+1})$ for each i.
- The source and the target of a path $\omega = e_1 \cdots e_n \in E^*$ $(n \geq 1)$ are defined as $\mathsf{s}(\omega) := \mathsf{s}(e_1)$ and $\mathsf{t}(\omega) := \mathsf{t}(e_n)$, respectively.
- A path $\omega \in E^*$ in Γ is closed if $\mathsf{s}(\omega) = \mathsf{t}(\omega)$ (or $\omega = \varepsilon$).

G-automata (3/3) — Formal definition

Definition ((non-deterministic) G-automata)

Let G be a group and Σ be a finite alphabet.

- A G-automaton is a 5-tuple $A = (\Gamma, \ell_G, \ell_\Sigma, p_{\mathrm{init}}, p_{\mathrm{ter}})$, where
  - $\Gamma = (V, E, \mathsf{s}, \mathsf{t})$ is a finite graph,
  - $\ell_G \colon E \to G$ and $\ell_\Sigma \colon E \to \Sigma \cup \{\varepsilon\}$ are edge-labeling functions, and
  - $p_{\mathrm{init}}, p_{\mathrm{ter}} \in V$ are initial vertex and terminal vertex, respectively.

  (Note that $(\Gamma, \ell_\Sigma, p_{\mathrm{init}}, p_{\mathrm{ter}})$ is a usual NFA)
- Edge-labeling functions are naturally extended to $\ell_G \colon E^* \to G$ and $\ell_\Sigma \colon E^* \to \Sigma^*$.
- A path $\omega \in E^*$ in Γ is an accepting path in A if

$$\mathsf{s}(\omega) = p_{\mathrm{init}}, \quad \mathsf{t}(\omega) = p_{\mathrm{ter}}, \quad \ell_G(\omega) = 1_G.$$

- The language $\mathrm{L}(A)$ accepted by a G-automaton A is defined by

$$\mathrm{L}(A) = \{\, \ell_\Sigma(\omega) \in \Sigma^* \mid \omega \text{ is an accepting path in } A \,\}.$$

The language class defined by G-automata

Theorem (Kambites, 2009)

The class $\mathcal{L}(G)$ of languages accepted by some G-automata forms a rational cone, i.e., it is closed under inverse homomorphism, homomorphism, and intersection with a regular language.
In particular, whether a word problem $\mathrm{WP}_\rho(H)$ is accepted by a G-automaton does not depend on the choice of ρ.

Examples

- For the trivial group $\{1\}$, $\mathcal{L}(\{1\}) =$ REG. (regular languages)
- If F is a free group of rank ≥ 2, $\mathcal{L}(F) =$ CFL. (context-free languages) (Chomsky–Schützenberger (1963), Corson (2005), Kambites (2009))

Re-interpretation of the known results

Theorem (Anīsīmov, 1972)

For a finitely generated group H, the following are equivalent:

1. The word problem $\mathrm{WP}(H)$ of H is **accepted by a $\{1\}$-automaton**.
2. H is a finite group (= virtually $\{1\}$).

Theorem (Muller–Schupp (1983) $+$ Dunwoody (1985))

For a finitely generated group H, the following are equivalent:

1. The word problem $\mathrm{WP}(H)$ of H is **accepted by a F-automaton**.
2. H is a virtually free group (i.e., has a finite index free subgroup).

Gilman posed a “commutative analog” of Muller–Schupp theorem: what is the class of groups whose word problem are accepted by $\mathbb{Z}^n$-automata?

Elder-Kambites-Ostheimer Theorem

Theorem (Elder-Kambites-Ostheimer, 2008)

Let H be a finitely generated group.
If the word problem $\mathrm{WP}(H)$ of H is accepted by a $\mathbb{Z}^n$-automaton A, then for some $m \leq n$, $\mathbb{Z}^m$ is a finite index subgroup of H.

However, their proof is somewhat complicated and:

- written in terms of geometric group theory, and
- depends on a deep theorem due to Gromov.

The rest of the talk, I give an elementary and purely combinatorial proof of their theorem.

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Proof (0/4): Outline

Theorem (Y., 2023)

If the word problem $\mathrm{WP}_\rho(H)$ of a fin. gen. group H is accepted by a G-automaton for an abelian group G, then there exists a surj. group hom. f from a subgroup $G_0 \subseteq G$ to a finite index subgroup $H_0 \subseteq H$.

1. Show that there are only finitely many minimal accepting paths.
2. For each min. acc. path μ and each vertex p, define a monoid $M(\mu, p)$ consisting of closed paths in A.
3. Show that each $M(\mu, p)$ induces a surj. homomorphism $f_{\mu,p}\colon G(\mu, p) \to H(\mu, p)$ from $G(\mu, p) \subseteq G$ onto $H(\mu, p) \subseteq H$.
4. At least one of $H(\mu, p)$'s has finite index in H by using the following:

Theorem (B. H. Neumann's lemma)

Let H be a group, $H_1, \ldots, H_n \subseteq H$ be subgr'ps, and $a_1, b_1, \ldots, a_n, b_n \in H$. If $H = \bigcup_{i=1}^{n} a_i H_i b_i$, then at least one of H_i's has finite index in H.

Proof (1/4): Notation for (scattered) subword

Let Σ be a finite alphabet and $u, v \in \Sigma^*$.

$$u \sqsubseteq v \quad :\Longleftrightarrow \quad \exists x, y \in \Sigma^* \; [xuy = v] \qquad \text{(subword)}$$

$$u \sqsubseteq_{\mathrm{sc}} v \quad :\Longleftrightarrow \quad \begin{matrix} \exists u_1, \ldots, u_n \in \Sigma^*, \\ \exists v_0, v_1, \ldots, v_n \in \Sigma^* \end{matrix} \begin{bmatrix} u = u_1 \; u_2 \; \cdots \; u_n, \\ v = v_0 u_1 v_1 u_2 v_2 \cdots v_{n-1} u_n v_n \end{bmatrix}$$

(scattered subword)

Remark

The both relations $\sqsubseteq$ and $\sqsubseteq_{\mathrm{sc}}$ are partial orders on Σ^*.

Proof (1/4): Minimal accepting paths

Definition

An accepting path $\alpha \in E^*$ is minimal if α is minimal with respect to $\sqsubseteq_{\mathrm{sc}}$ among the accepting paths.

Theorem (Higman's lemma)

The order $\sqsubseteq_{\mathrm{sc}}$ on E^* is a well-quasi-order, i.e., for every infinite sequence $\omega_1, \omega_2, \ldots \in E^*$, there are some $i < j$ such that $\omega_i \sqsubseteq_{\mathrm{sc}} \omega_j$.

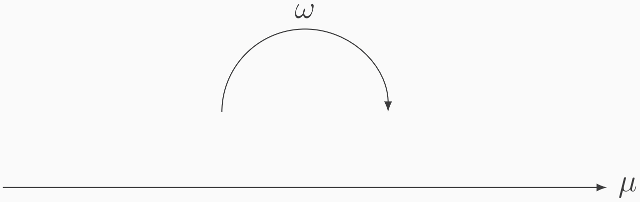
Corollary

There are only finitely many minimal accepting paths.
For every accepting path $\alpha \in E^*$, there exists a minimal accepting path $\mu \in E^*$ such that $\mu \sqsubseteq_{\mathrm{sc}} \alpha$.

Proof (2/4): Pumpable paths and the definition of $M(\mu, p)$

Definition

Let $\mu = e_1 \cdots e_n \in E^*$ ($e_i \in E$) be a minimal accepting path.
A path $\omega \in E^*$ is pumpable in μ if there is an accepting path
$\alpha = \alpha_0 e_1 \alpha_1 \cdots e_n \alpha_n \sqsupseteq_{\text{sc}} \mu$ such that $\omega \sqsubseteq \alpha_j$ for some j.



Remark

In above definition, each α_i is a closed path and
$\ell_G(\alpha_0) + \cdots + \ell_G(\alpha_n) = 0_G$ since G is commutative.

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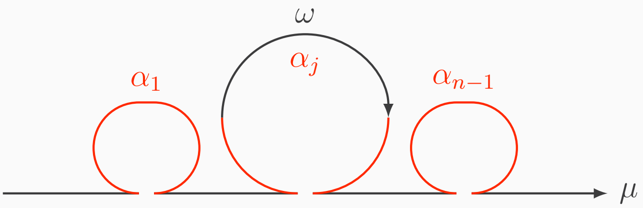

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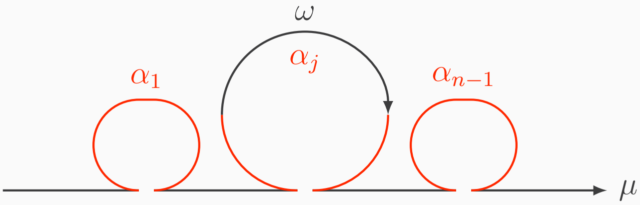

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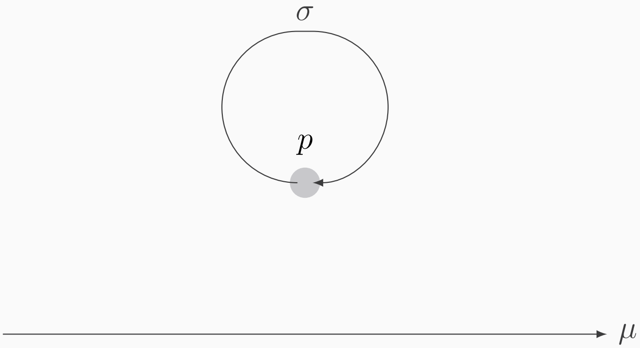
$$M(\mu, p) := \{\, \sigma \in E^* \mid \sigma \text{ is a } \textbf{closed} \text{ path pumpable in } \mu \text{ s.t. } \mathsf{s}(\sigma) = p \,\}$$



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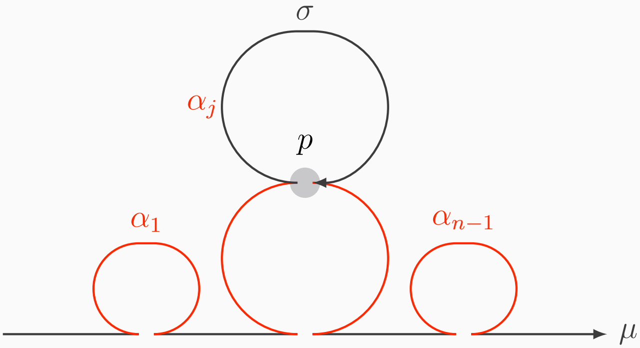


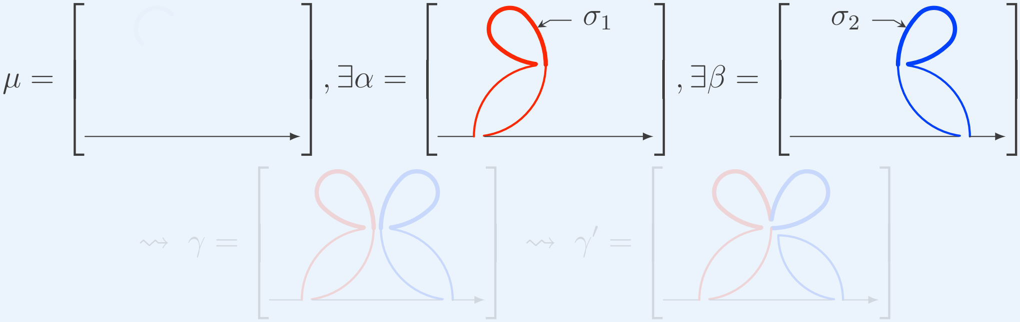
Proof (2/4): $M(\mu, p)$ forms a monoid

Lemma

Each $M(\mu, p)$ forms a monoid w.r.t. the concatenation.

Proof.

Let $\sigma_1, \sigma_2 \in M(\mu, p)$.



Thus $\sigma_1\sigma_2 \in M(\mu, p)$. □

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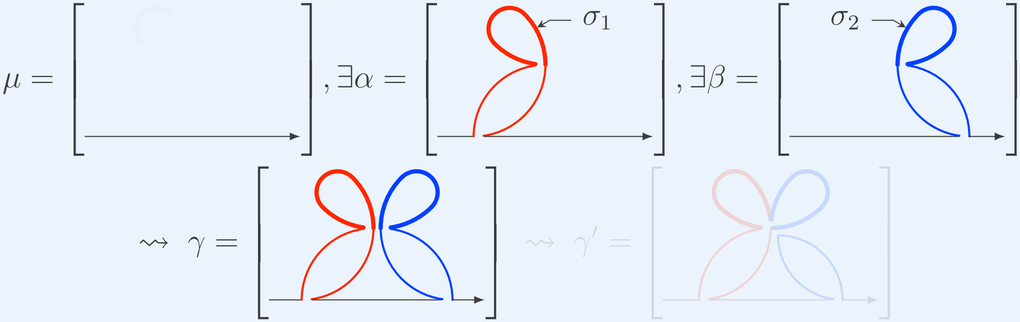

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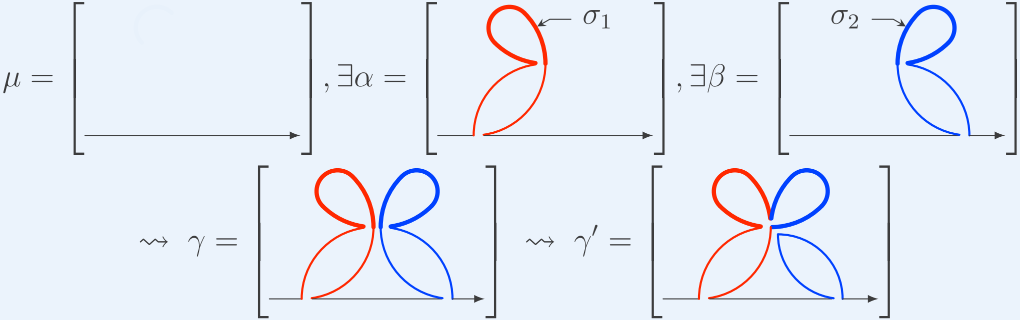

Thus $\sigma_1 \sigma_2 \in M(\mu, p)$. $\square$

Proof (3/4): Construction of $f_{\mu,p}\colon G(\mu,p) \to H(\mu,p)$

Take a natural surj. monoid hom. $\rho \circ \ell_\Sigma =: \varphi_{\mu,p}\colon M(\mu,p) \to \rho(\ell_\Sigma(M(\mu,p)))$. Then the function $\bar{\varphi}_{\mu,p}\colon \ell_G(M(\mu,p)) \to \rho(\ell_\Sigma(M(\mu,p)))$ is well-defined:

Lemma

Let two paths $\omega, \omega' \in E^*$ be subwords of accepting paths s.t. $\mathsf{s}(\omega) = \mathsf{s}(\omega')$ and $\mathsf{t}(\omega) = \mathsf{t}(\omega')$. Then $\ell_G(\omega) = \ell_G(\omega')$ implies $\rho(\ell_\Sigma(\omega)) = \rho(\ell_\Sigma(\omega'))$.

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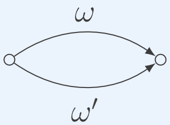

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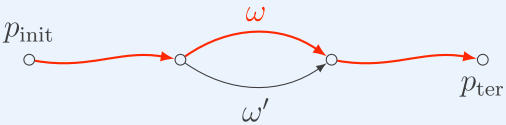

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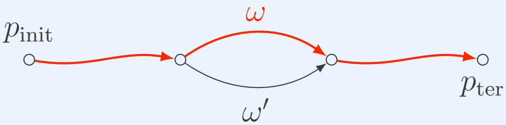

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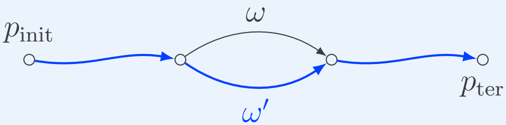

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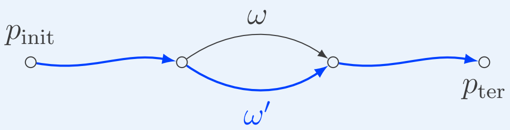

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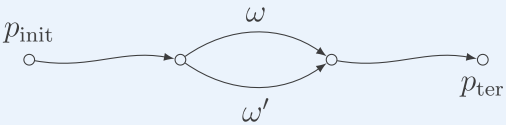
Then the function $\bar{\varphi}_{\mu,p} \colon \ell_G(M(\mu,p)) \to \rho(\ell_\Sigma(M(\mu,p)))$ is well-defined:

Lemma

Let two paths $\omega, \omega' \in E^*$ be subwords of accepting paths s.t. $\mathsf{s}(\omega) = \mathsf{s}(\omega')$ and $\mathsf{t}(\omega) = \mathsf{t}(\omega')$. Then $\ell_G(\omega) = \ell_G(\omega')$ implies $\rho(\ell_\Sigma(\omega)) = \rho(\ell_\Sigma(\omega'))$.

Proof.



$\rightsquigarrow \rho(\ell_\Sigma(\to \curvearrowright \to)) = 1_H = \rho(\ell_\Sigma(\to \circlearrowleft \to)) \quad \rightsquigarrow \rho(\ell_\Sigma(\overset{\omega}{\curvearrowright})) = \rho(\ell_\Sigma(\underset{\omega'}{\circlearrowleft})).$ $\square$

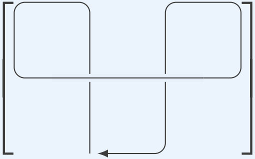
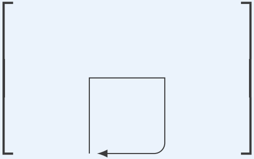
Defining $G(\mu,p) := \langle \ell_G(M(\mu,p)) \rangle \subseteq G$, $H(\mu,p) := \langle \rho(\ell_\Sigma(M(\mu,p))) \rangle \subseteq H$, we have a surjective group homomorphism $f_{\mu,p} \colon G(\mu,p) \to H(\mu,p)$.

Lemma

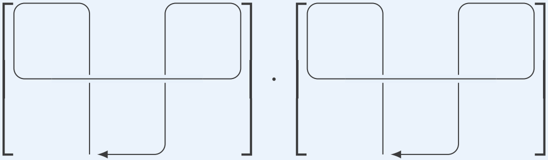
If $\tau \sqsubseteq_{\mathrm{sc}} \sigma \in M(\mu,p)$ and τ is a closed path s.t. $\mathsf{s}(\tau) = p$, then $\tau \in M(\mu,p)$.

Proof.

Suppose

$\sigma =$  , $\tau =$ 

Since $M(\mu,p)$ is a monoid, $\sigma^2 \in M(\mu,p)$.

$\sigma^2 =$ 

□

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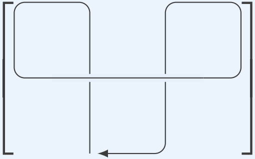
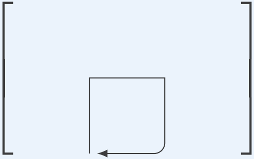
$\sigma =$  , $\tau =$ 

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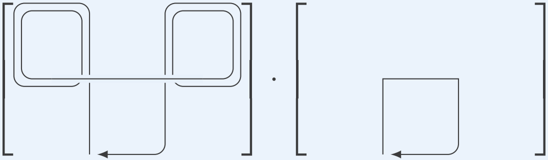


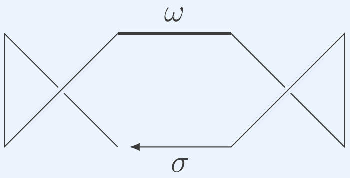
□

Proof (4/4): $(H(\mu, p))_{\mu,p}$ satisfy the hypo. of Neumann's lem. (2)

Corollary

If $\omega \sqsubseteq \sigma \in M(\mu, p)$, then $\omega_1 \omega \omega_2 \in M(\mu, p)$ for some $\omega_1, \omega_2 \in E^{<|V|}$.

Proof.



$\in M(\mu, p)$

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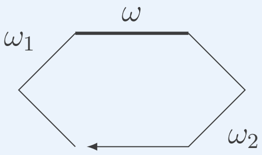

$\in M(\mu, p)$

□

Proof (4/4): $(H(\mu,p))_{\mu,p}$ satisfy the hypo. of Neumann's lem. (2)

Proposition

$$H = \bigcup \{\, h_1^{-1} H(\mu,p) h_2^{-1} \mid \mu \text{ is a min. acc. path in } A,\ p \in V, h_1, h_2 \in \rho(\Sigma^{<|V|}) \,\}.$$

Proof.

Let $N := 1 + \max\{\, |\mu| \mid \mu \in E^* \text{ is a min. acc. path} \,\}$.
Let $h = \rho(v) \in H$ ($v \in \Sigma^*$). There is some $\bar{v} \in \Sigma^*$ such that $\rho(\bar{v}) = \rho(v)^{-1}$.
Since $(v\bar{v})^N \in \mathrm{WP}_\rho(H) = \mathrm{L}(A)$, there is an acc. path $\alpha = \omega_1\bar{\omega}_1 \cdots \omega_N\bar{\omega}_N$ such that $\ell_\Sigma(\omega_i) = v, \ell_\Sigma(\bar{\omega}_i) = \bar{v}$.
Taking a min. acc. path $\mu = e_1 \cdots e_n \sqsubseteq_{\mathrm{sc}} \alpha$, we have $\alpha = \alpha_0 e_1 \alpha_1 \cdots e_n \alpha_n$ ($\alpha_i \in E^*$).
By the definition of N, some ω_i is "disjoint" from μ, i.e., $\omega_i \sqsubseteq \alpha_j$ for some j.
Thus we have $\alpha'_j \omega_i \alpha''_j \in M(\mu,p)$ ($p = \mathsf{s}(\alpha_j)$) for some $\alpha'_j, \alpha''_j \in E^{<|V|}$, and $h = \rho(\ell_\Sigma(\omega_i)) \in \rho(\ell_\Sigma(\alpha'_j))^{-1} H(\mu,p) \rho(\ell_\Sigma(\alpha''_j))^{-1}$. □

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