Separating the Wholeness axioms

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Large cardinal axioms

- Large cardinals are means to gauge the strength of extensions of ZFC.
- Since the beginning of set theory, set theorists defined stronger notion of large cardinals (Inaccessible, Mahlo, Weakly compact, Measurable, Woodin, Supercompact, etc.)
- Large cardinals stronger than measurable cardinals are usually defined in terms of elementary embedding.

Elementary embedding

Definition

Let $M \subseteq V$ be a transitive class. A map $j \colon V \to M$ is elementary if for every formula $\phi(\vec{x})$ over the language $\{\in\}$,

$$\phi(\vec{a}) \leftrightarrow \phi^M(j(\vec{a})).$$

κ is a critical point of j if κ is the least ordinal moved by j, i.e., $j(\kappa) > \kappa$.

Reinhardt embedding

Reinhardt introduced the following 'eventual' form of a large cardinal axiom:

Definition

A cardinal κ is a Reinhardt cardinal if it is the critical point of $j \colon V \to V$.

An elementary embedding $j \colon V \to V$ is called a Reinhardt embedding.

Icarian fate of Reinhardt cardinals

However, Reinhardt cardinals cannot exist over ZFC:

Theorem (Kunen 1971, ZFC)

There is no elementary embedding $j\colon V_{\lambda+2} \to V_{\lambda+2}$. As a corollary, there is no elementary embedding $j\colon V \to V$.

(If we take $\lambda = \sup_{n<\omega} j^n(\kappa)$, then $j \restriction V_{\lambda+2}\colon V_{\lambda+2} \to V_{\lambda+2}$.)

(Non-in)consistent weakening of Reinhardtness

Set theorists studied the non-inconsistent weakening of Reinhardt cardinals:

Definition

1. $\mathrm{I}_3(\lambda)$: There is an elementary $j\colon V_\lambda \to V_\lambda$.
2. $\mathrm{I}_2(\lambda)$: There is an Σ_1 -elementary[†] $j\colon V_{\lambda+1} \to V_{\lambda+1}$.
3. $\mathrm{I}_1(\lambda)$: There is an elementary $j\colon V_{\lambda+1} \to V_{\lambda+1}$.
4. $\mathrm{I}_0(\lambda)$: There is an elementary $j\colon L(V_{\lambda+1}) \to L(V_{\lambda+1})$.

They are not known to be inconsistent over ZFC.

[†] A formula is Σ_1 if it takes the form $\exists x \phi(x)$, where every quantifier in ϕ is bounded.

Other weakening

The obvious weakening is Reinhardt embedding without Choice.
The consistency of ZF with $j\colon V \to V$ is yet to be known, but

Theorem (Schultzenberg 2020)

If $\mathrm{ZFC} + \mathrm{I}_0(\lambda)$ *is consistent, then so is* $\mathrm{ZF} + (j\colon V_{\lambda+2} \to V_{\lambda+2})$.

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We can also consider Reinhardtness over a weaker theory, like ZFC without Power set:

Theorem (Matthews 2023)

$\mathsf{ZFC} + \mathrm{I}_1(\lambda)$ *proves the consistency of* $\mathsf{ZFC}^- + \exists j\colon V \to V$*.*

Here ZFC^- is a technical variant of 'ZFC without Power set.'

Formulating a Reinhardt embedding

An elementary embedding $j\colon V \to V$ is a proper class and not a set. That is, we cannot quantify over j.
To formulate j over ZFC, let us take the following approach:

Definition

ZFC_j is the theory over the language $\{\in, j\}$ with the following axioms:

1. Usual axioms of ZFC,
2. Axiom schema of Separation and Replacement are allowed for formulas over $\{\in, j\}$.
3. $j\colon V \to V$ is elementary: For every formula $\phi(\vec{x})$ over the language $\{\in\}$, we have

$$\phi(\vec{x}) \leftrightarrow \phi(j(\vec{x})).$$

The Wholeness axiom

Corazza introduced the Wholeness axiom by restricting Replacement to formulas over $\{\in\}$:

Definition

WA is the combination of the following statement:

1. Axiom schema of Separation for formulas over $\{\in, j\}$.
2. $j\colon V \to V$ is elementary.

$\mathrm{I}_3(\lambda)$ proves the consistency of WA; In fact, if $\mathrm{I}_3(\lambda)$ holds, then V_λ is a model of ZFC + WA.

Weaker variants of WA

Definition

A formula $\phi(x)$ is Σ^j_n if it takes of the form

$$\exists v_0 \forall v_1 \cdots \mathsf{Q} x_{v-1} \psi(v_0, v_1, \cdots, v_{n-1}, x),$$

where ψ is a formula over the language $\{\in, j\}$ in which every quantifier is bounded.
If ψ does not mention j, then we say ϕ is Σ_n.

Definition

WA_n is obtained from WA by restricting Separation schema to Σ^j_n-formulas.

Theorem (Hamkins 1999)

WA_0 *does not prove* WA_1.

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Theorem (J.)

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My proof does not take this form.

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1. Since there are finitely many formulas, there is n such that every formula of the fragment is Σ_n.
2. ZFC can define the truth predicate $\models_{\Sigma_n}$ for Σ_n-formulas.[†]
3. By the reflection principle, we can find α such that V_α respects $\models_{\Sigma_n}$. Hence V_α satisfies the fragment we fixed.

[†] In fact, KP suffices.

We want to mimic a similar argument to prove the consistency of $\mathsf{ZFC}+\mathsf{WA}_0$.
To do this, we must define a truth predicate that can capture every axiom of $\mathsf{ZFC}+\mathsf{WA}_0$.

Lemma ($\mathsf{ZFC}+\mathsf{WA}_0$)

Let $j\colon V\to V$ be the elementary embedding. If κ is the least ordinal moved by j, and if $\phi(x)$ is a formula over $\{\in\}$, then

$$\forall x\in V_\kappa[\phi(x)\leftrightarrow V_\kappa\vDash\phi(x)].$$

In other words, V_κ is an 'elementary substructure' of V.

Lemma (ZFC + WA_0)

Let j and κ be as before. If we let $\kappa_0 = \kappa$, $\kappa_{n+1} = j(\kappa_n)$, then

1. $\langle \kappa_n \mid n < \omega \rangle$ *is Σ^j_1-definable.*
2. $\langle \kappa_n \mid n < \omega \rangle$ *is cofinal over the class of all ordinals: That is, for every ordinal α there is $n < \omega$ such that $\alpha < \kappa_n$.*

These two lemma allow us to define a 'truth predicate' for formulas over $\{\in\}$:

Definition

$$\vDash_{\Sigma_\infty} \phi(x) \iff \exists n < \omega (x \in V_{\kappa_n} \wedge V_{\kappa_n} \vDash \phi(x)).$$

Extending the truth predicate

$\vDash_{\Sigma_\infty}$ covers every axiom of ZFC, but it is 'too simple' to cover WA_0 since $\vDash_{\Sigma_\infty}$ does not take any formulas with j.

Definition

A class of $\Delta_0^j(\Sigma_\infty)$ formulas is the least class of formulas containing formulas in $\{\in\}$ closed under

1. Boolean connectives $(\wedge, \vee, \neg, \to)$, and
2. Bounded quantifiers, which take of the form $\forall u \in j^n(x)$ or $\exists u \in j^n(x)$.

We can define the truth predicate $\vDash_{\Delta_0^j(\Sigma_\infty)}$ for $\Delta_0^j(\Sigma_\infty)$ formulas in a Σ_1^j way, in which we will omit the details.

The unreachable

Recall that we are mimicking the following argument:

1. Since there are finitely many formulas, there is n such that every formula of the fragment is Σ_n.
2. ZFC can define the truth predicate $\vDash_{\Sigma_n}$ for Σ_n-formulas.
3. By the reflection principle, we can find α such that V_α respects $\vDash_{\Sigma_n}$. Hence V_α satisfies the fragment we fixed.

The unreachable

Recall that we are mimicking the following argument:

1. Since $\vDash_{\Sigma_\infty}$ is Σ^j_1-definable, every axiom of $\mathsf{ZFC} + \mathsf{WA}_0$ is finitely axiomatizable.
2. We can define $\vDash_{\Delta^j_0(\Sigma_\infty)}$.
3. Do we have a reflection argument?

The latter step won't work because we do not have Replacement for j-formulas.

Strong soundness: What shines the darkness

To get around the issue, we need a proof-theoretic tool:

Definition

Let term_V be the class of all terms generated from constant symbols $\{c_x \mid x \in V\}$ corresponding the class of all sets with a function symbol j.
Let Form_V be the class of all formulas over $\{\in, \mathrm{j}\}$, with terms from term_V.
For a set X of sentences over $\{\in, \mathrm{j}\}$, let S_V^X be the least class containing X and closed under subformulas, term substitution, and Boolean combinations.

Definition

Let X be a set of sentences over $\{\in, \mathrm{j}\}$. A class function $T\colon \mathrm{Form}_V \cup \mathrm{S}^X_V \to V$ is a weak class model for X if

1. $T(\mathrm{j}(t)) = j(T(t))$ for $t \in \mathrm{term}_V$.
2. T respects the Tarskian truth definition, i.e.,
   - For terms s, s' and t, t', if $T(s) = T(s')$, $T(t) = T(t')$, then $T(\ulcorner s = t \urcorner) = T(\ulcorner s' = t' \urcorner)$ and $T(\ulcorner s \in t \urcorner) = T(\ulcorner s' \in t' \urcorner)$.
   - $T(\ulcorner \neg\sigma \urcorner) = 1 - T(\ulcorner \sigma \urcorner)$.
   - If $\circ$ is a logical connective, then $T(\ulcorner \phi \circ \psi \urcorner) = 1$ if and only if $T(\ulcorner \phi \urcorner) \circ T(\ulcorner \psi \urcorner) = 1$.
   - If Q is a quantifier, then $T(\ulcorner \mathrm{Q}x\phi(x) \urcorner) = 1$ if and only if $\mathrm{Q}x[T(\ulcorner \phi(x) \urcorner) = 1]$ holds.*

*It applies only when $\ulcorner \mathrm{Q}x\phi(x) \urcorner \in \mathrm{S}^X_V$.

The main feature of a weak class model is that it evaluates the truth of a class of formulas even if the class is not closed under quantifiers.
The following lemma says a weak class model is enough to establish the consistency:

Lemma (Strong Soundness, $\mathsf{ZFC} + \mathsf{WA}_1$)

If there is a Π^j_1-definable weak class model for X, then X is consistent.

We can construct a Π^j_1-definable class model of $\mathsf{ZFC} + \mathsf{WA}_0$ from $\vDash_{\Delta^j_0(\Sigma_\infty)}$.

Questions

Thank you!