Implicational tonoids, embeddability, and representations

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Outline

1. Introduction: abstract / background / motivation
2. Preliminaries: tonoids / implicational tonoid matrices / weakly implicative matrices
3. Representation I: frames / embeddability and representation
4. Assertional implicational tonoid algebras
5. Representation II: frames / embeddability and representation

1. Introduction: Abstract

- Yang and Dunn (2021) introduced implicational tonoid logics as logics combining two classes of generalized logics, one of which is the class of weakly implicative logics introduced by Cintula and the other of which is the class of gaggle logics introduced by Dunn.

- We extend this investigation to representations.

1. Introduction: Abstract

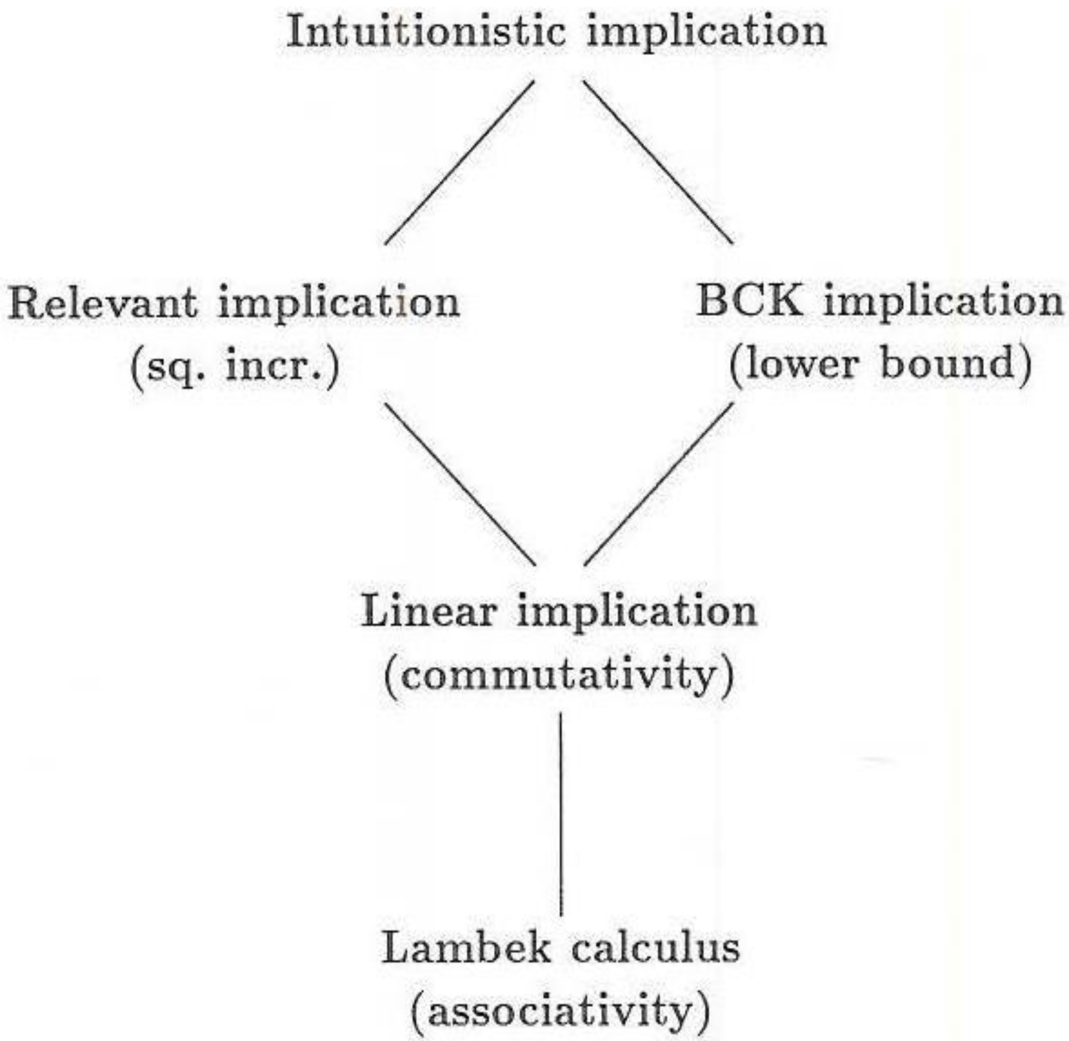
More precisely,

- First, as preliminaries we review implicational tonoid matrices. (Sect 2)
- Second, we introduce their representations. We in particular consider embeddability property for implicational tonoid matrices. (Sect 3)
- Third, we generalize these to assertional implicational tonoid algebras and their representations. (Sects 4-5)

1. Introduction: Background

1. Implication and partial order

- One important trend in alternative logics is to introduce abstract logics with more general structures.
- In this tradition, the logical connective "implication" is very important in the sense that systems of logic are often distinguished by the properties of their implications (see e.g. the Kite below).
- These implications all share at least the underlying properties of reflexivity (A implies A is provable) and transitivity (similarly).
- This suggests an abstraction based on preordered sets.
- When A is an algebra, we want an equivalence relation $\equiv$ to be a congruence.
- Preordered sets can be regarded as partially ordered sets by defining an equivalence relation $\equiv$.


Intuitionistic implication
Relevant implication
(sq. incr.)
BCK implication
(lower bound)
Linear implication
(commutativity)
Lambek calculus
(associativity)

1. Introduction: Background

2. Abstract algebras/matrices based on partially ordered sets (briefly, posets)

Trend 1

- Implicative algebras, (for logics based on posets with the greatest element 1 as the sole designated element, by Rasiowa (1974))
- Weakly implicative matrices, a generalization of implicative algebras, (for logics based on posets with operations satisfying congruence and with D, a set of designated elements, by Cintula (2006))

Trend 2

- Gaggles, generalized Galois logics, (for logics based on distributive lattices with operations satisfying abstract Galois connection properties, by Dunn (1991))
- Partial gaggles, a generalization of gaggles, (for logics based on posets with operations satisfying abstract Galois connection properties, by Dunn (1993))
- Tonoids, a generalization of partial gaggles, (for logics based on posets with operations with tonic properties, by Dunn (1993))

1. Introduction: Motivation, Aim

- Motivation

(1) Abstract logics with more general structures combining weakly implicative matrices and tonoids have been introduced with the title "Implicational tonoid logics" (Yang & Dunn 2021). However,

(2) Related representations have not yet been investigated.

- The *aim* of this talk

(1) To introduce related representations.

(2) To generalize the matrices to their corresponding algebras and their representations.

2. Preliminaries

- A tonic language is an algebraic language *L* with a tonicity map **ttyp** which maps every operation # of arity n > 0 to its tonic type **ttyp**(#) = (s_1, ..., s_n), where each s_i is + (isotone) or - (antitone).

- #($\underline{x}$,y_i) denotes the application of # to *n* arguments, where $\underline{x}$ is the sequence of arguments of # excepting its i-th argument and y is the i-th argument of #.

- A tonoid is a poset with a set of *n*-ary operations being isotone or antitone on each *i*-th, $i \leq n$, argument (Dunn (1993))

2. Preliminaries

Examples of tonicity

(1) Isotone:

a) if $x \leq y$, then $(z \rightarrow x) \leq (z \rightarrow y)$

b) if $x \leq y$, then $(z * x) \leq (z * y)$, $(x * z) \leq (y * z)$,

(2) Antitone:

a) if $x \leq y$, then $(y \rightarrow z) \leq (x \rightarrow z)$

b) if $x \leq y$, then $-y \leq -x$

2. Preliminaries

Definition 2.1 (Implicational tonoid matrices)

Let L be a tonic language, such that $(\Rightarrow; 2) \in L$ and **ttyp**$(\Rightarrow) = (-,+)$. A structure $\boldsymbol{A} = (A, \leq, \Rightarrow, \{\#\}; D)$ is said to be an implicational tonoid matrix if

(1) $(A, \leq, \{\#\})$ is a tonoid.

(2) $x \leq y$ iff $x \Rightarrow y \in D$, for $x, y \in A$ and $D \subseteq A$.

Definition 2.2 (Weakly implicative matrices)

Let L be an algebraic language, such that $(\Rightarrow; 2) \in L$. A structure $\boldsymbol{A} = (A, \leq, \Rightarrow, \{\#\}; D)$ is said to be a weakly implicative matrix if $(A, \leq)$ is a poset satisfying (2) in Definition 2.1 and the following.

($Cong_{\#}{}^{i}$, congruence) if $x \Leftrightarrow y \in D$, then $\#(\underline{z},x_i) \Rightarrow \#(\underline{z},y_i) \in D$.

Theorem 2.3 Implicational tonoid matrices are weakly implicative matrices.

3. Representation I: Preliminaries

- A labeled language is a tonic language L equipped with a labeling map **ltyp** which maps every operator # of arity n > 0 to its labeled type **ltyp**(#) ∈ {□, ◇}.

Definition 3.1 (Implicational tonoid matrix in labeled language)

Let L be a labeled language, such that (⇒; 2) ∈ L, **ltyp**(⇒) = □, and **ttyp**(⇒) = (-,+). A structure ***A*** is said to be an implicational tonoid matrix iff Definition 2.1, where **ltyp**(#) ∈ {□, ◇}, holds in ***A*** .

- We use the notations \`±,' \` ⋔' if we need not specify one of isotonicity + and antitonicity -, and one of membership ∈ and non-membership ∉, respectively.

3. Routley-Meyer-style frames

Definition 3.1

1) (Implicational Routley-Meyer-style frame) For an implicational partially ordered set matrix ***A*** = (A, $\leq$, $\Rightarrow$, D), an *implicational Routley-Meyer-style frame* (briefly, R-$M_{\Rightarrow}$ frame) for ***A*** is meant a structure ***F*** = (F, $\leq$, $R_{\Rightarrow}$, D), where (F, $\leq$) is a partially ordered set and $R_{\Rightarrow}$ satisfies the postulates below: for all a, b $\in$ F,

($p_{\leq}$) a $\leq$ b iff there is c $\in$ D such that $R_{\Rightarrow}$(a,b;c), briefly $R_{\Rightarrow}$(a,b;D).

(p_M) for all a, b, c $\in$ F,

- $R_{\Rightarrow}$(a,b;c) and a' $\leq$ a imply $R_{\Rightarrow}$(a',b;c),
- $R_{\Rightarrow}$(a,b;c) and b $\leq$ b' imply $R_{\Rightarrow}$(a,b';c),
- $R_{\Rightarrow}$(a,b;c) and c' $\leq$ c imply $R_{\Rightarrow}$(a,b;c').

3. Routley-Meyer-style frames

Definition 3.1

2) (Implicational tonoid R-M frames, briefly T-R-M$_{\Rightarrow}$ frames) For an implicational tonoid matrix ***T*** = (A, ≤, ⇒, {#}, D), a *T-R-M*$_{\Rightarrow}$ *frame* for ***T*** is meant a structure ***F*** = (F, ≤, $R_{\Rightarrow}$, {$R_{\#}$}, D), where (F, ≤, $R_{\Rightarrow}$, D) is an R-M$_{\Rightarrow}$ frame and $R_{\#}$ satisfies the definitions and postulates below:

(df1) $R_{\#\Rightarrow}((\underline{a},b_i),c;v)$ iff there is x s. t. $R_{\#}(\underline{a},b_i;x)$ and $R_{\Rightarrow}(x,c;v)$.

(pTon$_{\#\Diamond+}$) if $R\#_{\Diamond(\pm)+\Rightarrow}((\underline{c},a_i),v;D)$, there is x s. t. $R\#_{\Diamond(\pm)+}(\underline{c},x_i;v)$ and $a \leq x$.

(pTon$_{\#\Diamond-}$) if $R\#_{\Diamond(\pm)-\Rightarrow}((\underline{c},a_i),v;D)$, there is x s. t. $R\#_{\Diamond(\pm)-}(\underline{c},x_i;v)$ and $x \leq a$.

(df1') $R_{\#\Rightarrow}(c,(\underline{a},b_i);v)$ iff there is x s. t. $R\#(\underline{a},b_i;x)$ and $R_{\Rightarrow}(c,x;v)$.

(pTon$_{\#\Box+}$) if $R\#_{\Box(\pm)+\Rightarrow}(v,(\underline{c},a_i);D)$, there is x s. t. $R\#_{\Box(\pm)+}(\underline{c},x_i;v)$ and $x \leq a$.

(pTon$_{\#\Box-}$) if $R\#_{\Box(\pm)-\Rightarrow}(v,(\underline{c},a_i);D)$, there is x s. t. $R\#_{\Box(\pm)-}(\underline{c},x_i;v)$ and $a \leq x$.

3. Matrices

We then show that implicational tonoid matrices can be defined as matrices based on T-R-$M_{\Rightarrow}$ frames.

Proposition 3.2 (Let ***F*** = (F, ≤, $R_{\Rightarrow}$,{$R_{\#}$}, D) be a T-R-$M_{\Rightarrow}$ frame. We can define an implicational tonoid matrix $\boldsymbol{F}^{+}$ = (F^{+}, $\leq^{+}$, $\Rightarrow^{+}$, {$\#^{+}$}, D^{+}) on subsets of F, where $\Rightarrow^{+}$ and $\#^{+}$'s preserve the labeled and tonic types of ⇒ in $R_{\Rightarrow}$ and # in $R_{\#}$.

3. Canonical frames

- For every implicational tonoid matrix $\boldsymbol{T} = (T, \leq, \Rightarrow, \{\#\}; D)$, we define the canonical structure $\boldsymbol{F}^{can} = (F^{can}, \leq^{can}, R^{can}_{\Rightarrow}, \{R^{can}_{\#}\}; D^{can})$ on $\boldsymbol{T}$ as follows:
- Canonical T-R-M$\Rightarrow$ frame:

- F^{can} is the set of all cones on $(T, \leq)$;
- $\leq^{can}$ is the inclusion relation between elements of F^{can};
- $D^{can} = \{D\}$;
- R^{can}'s are defined as follows:

3. Canonical frames

- $R^{can}_{\#\Diamond(\pm)+}(\underline{A},B_i;C)$ iff for all $\underline{x}$, y ∈ T, if $\underline{x}$ ⋔ $\underline{A}$ and y ∈ B, then $\#^n(\underline{x},y_i)$ ∈ C;

- $R^{can}_{\#\Diamond(\pm)-}(\underline{A},B_i;C)$ iff for all $\underline{x}$, y ∈ T, if $\underline{x}$ ⋔ $\underline{A}$ and y ∉ B, then $\#^n(\underline{x},y_i)$ ∈ C.

- $R^{can}_{\#\Box(\pm)+}(\underline{A},B_i;C)$ iff for all $\underline{x}$, y ∈ T, if $\#^n(\underline{x},y_i)$ ∈ C and $\underline{x}$ ⋔ $\underline{A}$, then y ∈ B;

- $R^{can}_{\#\Box(\pm)-}(\underline{A},B_i;C)$ iff for all $\underline{x}$, y ∈ T, if $\#^n(\underline{x},y_i)$ ∈ C and y ∈ B, then $\underline{x}$ ⋔ $\underline{A}$.

3. Canonical frames

Lemma 3.3 Let $\boldsymbol{F}^{can}$ be a canonical structure defined as above and a, b $\in$ T. If a $\Rightarrow$ b $\in$ D, then for each A $\in$ F^{can}, a $\in$ A implies b $\in$ A.

$\boldsymbol{F}^{can}$ is said to be *inductive* if the set-theoretic union of every non-empty chain in $\boldsymbol{F}^{can}$ belongs to $\boldsymbol{F}^{can}$.

Proposition 3.4 The canonically defined inductive structure $\boldsymbol{F}^{can} = (F^{can}, \leq^{can}, R^{can}{}_{\Rightarrow}, \{R^{can}{}_{\#}\}; D^{can})$ is indeed a T-R-$M_{\Rightarrow}$ frame.

3. Embeddabilty, representation

Theorem 3.5 (Embeddability) Every implicational tonoid matrix $\boldsymbol{T} = (T, \leq, \Rightarrow, \{\#\}; D)$ is embeddable into the implicational tonoid matrix $\boldsymbol{F}^{\mathrm{can}\,+}$.

Corollary 3.6 (Representation) Every implicational tonoid matrix $\boldsymbol{T} = (T, \leq, \Rightarrow, \{\#\}; D)$ is representable as an implicational tonoid matrix defined on a T-R-$M_{\Rightarrow}$ frame as in Proposition 3.2.

4. Assertional implicational tonoid algebras

Definition 4.1 (Assertional implicational tonoid algebras)

Let L be a tonic language, such that $(\Rightarrow; 2) \in L$ and **ttyp**$(\Rightarrow) = (-,+)$. A structure $\boldsymbol{A} = (A, \leq, \Rightarrow, \{\#\}; e)$ is said to be an assertional implicational tonoid algebra if

(1) $(A, \leq, \{\#\})$ is a tonoid.

(2) $x \leq y$ iff $e \leq x \Rightarrow y$, for $x, y \in A$.

Definition 4.2 (Assertional weakly implicative algebras)

Let L be an algebraic language, such that $(\Rightarrow; 2) \in L$. An assertional weakly implicative algebra is a weakly implicative matrices having e in place of D and satisfying $(\mathrm{Cong}_{\#}{}^{Ai})$ below instead of $(\mathrm{Cong}_{\#}{}^{i})$ above.

$(\mathrm{Cong}_{\#}{}^{Ai})$ if $e \leq x \Leftrightarrow y$, then $e \leq \#(\underline{z}, x_i) \Rightarrow \#(\underline{z}, y_i)$.

Theorem 4.3 Assertional implicational tonoid algebras are assertional weakly implicative algebras.

4. Assertional implicational tonoid algebras

The definition of an assertional implicational tonoid algebra can be refined in labeled language as follows.

Definition 4.4 (Assertional implicational tonoid algebras in labeled language) Let L be a labeled language, such that $(\Rightarrow; 2) \in L$ **ltyp**(⇒) = □ and **ttyp**(⇒) = (-,+). A structure ***A*** = (A, ≤, ⇒, {#}; e) is said to be an assertional implicational tonoid algebra iff Definition 4.1, where **ltyp**(#) ∈ {□, ◇}, holds in ***A*** .

5. Routley-Meyer-style frames

Let (P, $\leq$, e) be an assertional partially ordered set algebra, i.e., a partially ordered set with e $\in$ P.

Given an n+1-nary relation R on P, $R(a_1, \ldots, a_n; e)$ henceforth means that there exists c such that $e \leq c \in P$ and $R(a_1, \ldots, a_n; c)$.

As above, we further assume that tonicity and labeling maps **ttyp** and **ltyp** are also preserved for relations.

5. Routley-Meyer-style frames

Definition 5.1 (Assertional implicational tonoid relational frames, briefly T-R-$M_{e\Rightarrow}$ frames) For an assertional implicational tonoid algebra ***T*** = (A, ≤, ⇒, {#}, e), a *T-R-$M_{e\Rightarrow}$ frame* for ***T*** is meant a structure ***F*** = (F, ≤, $R_{\Rightarrow}$, {$R_{\#}$}, e), where (F, ≤, $R_{\Rightarrow}$, e) is an R-$M_{\Rightarrow}$ frame and $R_{\#}$ satisfies the definitions (df1), (df1') and the postulates below:

($p'Ton_{\#\Diamond+}$) if $R_{\#\Diamond(\pm)+\Rightarrow}((\underline{c},a_i),v;e)$, there is x s. t. $R_{\#\Diamond(\pm)+}(\underline{c},x_i;v)$ and $R_{\Rightarrow}(a,x;e)$.

($p'Ton_{\#\Diamond-}$) if $R_{\#\Diamond(\pm)-\Rightarrow}((\underline{c},a_i),v;e)$, there is x s. t. $R_{\#\Diamond(\pm)-}(\underline{c},x_i;v)$ and $R_{\Rightarrow}(x,a;e)$.

($p'Ton_{\#\Box+}$) if $R_{\#\Box(\pm)+\Rightarrow}(v,(\underline{c},a_i);e)$, there is x s. t. $R_{\#\Box(\pm)+}(\underline{c},x_i;v)$ and $R_{\Rightarrow}(x,a;e)$.

($p'Ton_{\#\Box-}$) if $R_{\#\Box(\pm)-\Rightarrow}(v,(\underline{c},a_i);e)$, there is x s. t. $R_{\#\Box(\pm)-}(\underline{c},x_i;v)$ and $R_{\Rightarrow}(a,x;e)$.

5. Algebras

We then show that implicational tonoid algebras can be defined as algebras based on T-R-$M_{e\Rightarrow}$ frames.

Proposition 5.2 Let $\boldsymbol{F}$ = (F, ≤, $R_{\Rightarrow}$, {$R_{\#}$}, e) be a T-R-$M_{e\Rightarrow}$ frame. We can define an assertional implicational tonoid algebra $\boldsymbol{F}^{+}$ = (F^{+}, $\leq^{+}$, $\Rightarrow^{+}$, {$\#^{+}$}, e^{+}) on subsets of F, where $\Rightarrow^{+}$ and $\#^{+}$'s preserve the labeled and tonic types of ⇒ in $R_{\Rightarrow}$ and # in $R_{\#}$.

5. Canonical frames

- The canonical (inductive) structure $\boldsymbol{F}^{can} = (F^{can}, \leq^{can}, R^{can}_{\Rightarrow}, \{R^{can}_{\#}\};\ e^{can})$ on an assertional implicational tonoid algebra $\boldsymbol{T} = (T, \leq, \Rightarrow, \{\#\}; e)$ is defined as in Section 3.1 except the definition of e^{can} below.

  $e^{can} = \{a \in T: e \leq a\}$

Lemma 5.3 Let $\boldsymbol{F}^{can}$ be a canonical structure defined as above and a, b $\in$ T. If $e \leq a \Rightarrow b$, then for each $A \in F^{can}$, $a \in A$ implies $b \in A$.

Proposition 5.4 The canonically defined inductive structure $\boldsymbol{F}^{can} = (F^{can}, \leq^{can}, R^{can}_{\Rightarrow}, \{R^{can}_{\#}\}; e^{can})$ is indeed a $T\text{-}R\text{-}M_{e\Rightarrow}$ frame.

5. Embeddabilty, representation

Theorem 5.5 (Embeddability) Every assertional implicational tonoid algebra $\boldsymbol{T} = (T, \leq, \Rightarrow, \{\#\}; e)$ is embeddable into the assertional implicational tonoid algebra $\boldsymbol{F}^{can\,+}$.

Corollary 5.6 (Representation) Every assertional implicational tonoid algebra $\boldsymbol{T} = (T, \leq, \Rightarrow, \{\#\}; e)$ is representable as an assertional implicational tonoid algebra defined on a T-R-$M_{e\Rightarrow}$ frame as in Proposition 5.1.

6. Concluding remarks

We first investigated implicational tonoid matrices and their representations and then generalized to assertional implicational tonoid algebras and their representations.

We may introduce some expansions of the results such as implicational partial gaggle matrices and their representations. Moreover, we may extend these results to their corresponding assertional algebras and their representations. But we leave this for another work.

Thank you!