An application of constructive dependent type theory to certified computation over the reals

Holger Thies (Kyoto University)
j.w.w. Michal Konečný (Aston U) and Sewon Park (Kyoto U)

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- Arithmetical operations are computed exactly, without any rounding errors.
- Realistic in the sense that programs can be executed on a computer.
- Many frameworks exist (iRRAM, AERN).

No Rounding errors

Rump's example

$$R(x, y) = (333.75 - x^2)y^6 + x^2(11x^2y^2 - 121y^4 - 2) + 5.5y^8 + \frac{x}{2y}$$

evaluated at $x = 77617$ and $y = 33096$.

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- Can be implemented in ERC frameworks directly
- Output up to any desired precision
- Simple mathematical proofs of correctness

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restr_sqrt x = -- restricted to 0.25 < x < 2
  limit $
    \n -> sqrt_approx_fast x n
```

Semi-decidable comparisons

Comparison is partial. Kleenean comparison used instead:

```
...> pi > 0
{?(prec 36): CertainTrue}

...> pi == pi
{?(prec 36): TrueOrFalse}

...> pi == pi + 2^(-100)
{?(prec 36): TrueOrFalse}

...> (pi == pi + 2^(-100)) ? (prec 1000)
CertainFalse
```

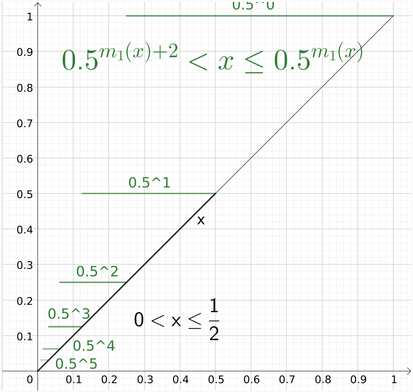
Multivalued selection

- Branching over semi-decidable comparison can be problematic
- `select` increases the precision until one of the Kleenans becomes `CertainTrue`.

```
max_nondeterministic x y =
 limit $ \n ->
   let e = 0.5^n in
   if select (x > y - e) (y > x - e)
        then x
        else y
```

Non-extensionality

```
magnitude1 x =
 integer $ fromJust $ List.findIndex id $ map test [0..]
 where
 test n = select (0.5^(n+2) < x) (x < 0.5^(n+1))
```



Certified exact real computation

Why certified exact real computation?
Limits, Non-determinism, etc. can easily go wrong.

Approaches to certified exact real computation:

- Program verification
  - ERC, Incone, ...
- Program extraction from (constructive) proofs
  - CorN, IFP, Minlog, **cAERN**, ...

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Michal Konečný, Sewon Park, and Holger Thies.
Axiomatic Reals and Certified Efficient Exact Real Computation.
WoLLIC 2021. Springer, Cham, 2021.

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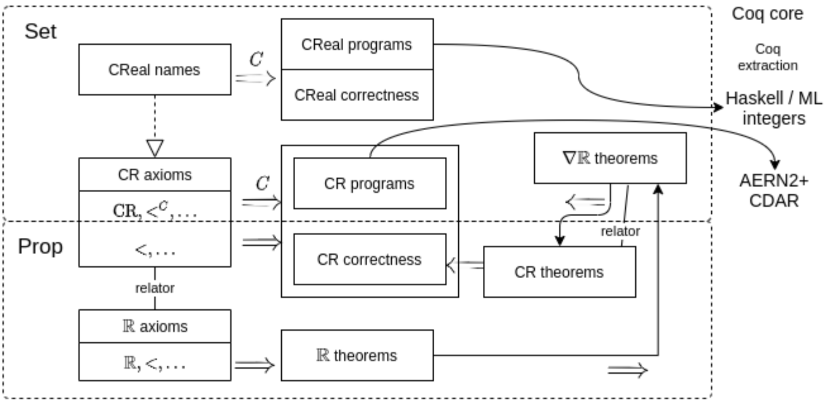
cAERN



Set
Coq core
CReal names
C
CReal programs
CReal correctness
Coq extraction
Haskell / ML integers
CR axioms
CR, $<^C, \ldots$
C
CR programs
$\nabla\mathbb{R}$ theorems
AERN2+ CDAR
relator
Prop
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- All our theory is implemented in the Coq proof assistant.
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We assume to work in a dependent type theory with

- Basic Types $0, 1, 2, \mathbb{N}, \mathbb{Z}$,
- An impredicative universe Prop of classical propositions closed under $\to, \wedge, \vee, \exists, \Pi$,
- A predicative universe Type with type constructors $\to, \times, +, \Sigma, \Pi$

Background

Prop is classical

- Law of excluded middle $\Pi(P : \text{Prop}).\ P \vee \neg P$
- Propositional extensionality

$$\Pi(P, Q : \text{Prop}).\ (P \leftrightarrow Q) \rightarrow P = Q$$

- Countable choice

$$\begin{gathered}\Pi(A : \text{Type}).\ \Pi(P : \mathbb{N} \rightarrow A \rightarrow \text{Prop}).\\ (\Pi(n : \mathbb{N}).\ \exists(x : X).\ P\, n\, x) \rightarrow \exists(f : \mathbb{N} \rightarrow A).\ \Pi(n : \mathbb{N}).\ P\, n\, (f\ n)\end{gathered}$$

- Functional Extensionality, Markov principle

Axiomatized Reals

Field axioms:

$$
\begin{aligned}
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&0 : R \\
&1 : R \\
&+ : R \to R \to R \\
&\quad\vdots
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$$R : \text{Type}$$

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$$\vdots$$

Classical order:

$$<: R \to R \to \text{Prop}$$

$$\Pi(x, y : R).\ x < y \lor x = y \lor x > y$$

$$\vdots$$

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For example, we assume semi-decidability of comparisons:

$$\Pi(x, y : R).\ \Sigma(t : \mathsf{K}).\ x < y \leftrightarrow t = \mathit{true}$$

Nondeterminism

Nondeterminsm monad: For any type X : *Type* we automatically get a type $\mathsf{M}X$: *Type*.

$$A \to \mathsf{M}B \Rightarrow \text{nondeterministic function from } A \text{ to } B$$

$$\mathsf{M}(A + B) \Rightarrow \text{nondeterministic decision if } A \text{ or } B$$

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For any two semi-decidable decisions $x, y : \mathsf{K}$, if promised that either of x or y holds classically, we can nondeterministically decide whether x holds or y holds:

$$\Pi(x, y : \mathsf{K}).\ (x = \mathsf{true} \lor y = \mathsf{true}) \to \mathsf{M}\big(x = \mathsf{true} + y = \mathsf{true}\big)$$

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Nondeterministic soft comparison:

$$\Pi(x, y : R).\ \Pi(n : \mathbb{N}).\ \mathsf{M}(x < y + 2^{-n} + y < x + 2^{-n})$$

Nondeterminsim and Limits

Consider the following three cases where we might want to compute a limit

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3. A sequence of nondeterministic reals converges to a nondeterministic point: $f : \mathbb{N} \to \mathsf{M}R \rightsquigarrow x : \mathsf{M}R$

Deterministic Limits

We assume constructive metric completeness:
Whenever we have a sequence $f : \mathbb{N} \to R$ that is fast Cauchy, i.e.,

$$\Pi(n, m : \mathbb{N}).\ -2^{-n-m} \leq f\ n - f\ m \leq 2^{-n-m}$$

there is a limit point of the sequence, i.e.

$$\Sigma(x : R).\ \Pi(n : \mathbb{N}).\ -2^{-n} \leq f\ n - x \leq 2^{-n}$$

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- $0.25 \leq 4^{-z}x \leq 1$ and $\sqrt{x} = 2^z\sqrt{4^{-z}x}$

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The sequence converges to a square root, thus we should be able to show

$$\Sigma(y : R).\ x \geq 0 \implies y \cdot y = x$$

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- Each $z \neq 0$ has exactly two square roots.
- There is no total, continuous mapping $\sqrt{} : \mathbb{C} \to \mathbb{C}$, the complex square root is inherently multivalued.
- However, the nondeterministic function computing any of the two square roots is computable.

Complex square root

We can reduce complex square roots to real square roots:

Let $z = a + ib$, then

$$\sqrt{\frac{\sqrt{a^2+b^2}+a}{2}} + i\,\mathrm{sgn}(b)\sqrt{\frac{\sqrt{a^2+b^2}-a}{2}}$$

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- $z \neq 0 \to \mathsf{M}(a < 0 + a > 0 + b < 0 + b > 0)$
- Nondeterministically choose one of the four cases and adapt the formula to the case:

$$z \neq 0 \to \mathsf{M}\Sigma(x : C).\ x \cdot x = z$$

Square root as limit

Given $z \in \mathbb{C}$, consider the following recursively defined (nondeterministic) sequence. For each n and given a previous approximation x_{n-1},

- Start by nondeterministically choosing one of the two cases $\|z\| \leq 2^{-2(n+2)}$ or $\|z\| > 0$

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Any such sequence is a Cauchy sequence converging against one of the square roots of z, thus we expect to nondeterministically have a limit point.

Nondeterministic dependent choice

- Suppose we have $x_0 : A$ and $f : \mathbb{N} \to A \to \mathrm{M}A$.
- We can get a nondeterministic sequence by repeatedly applying $x_{n+1} := f\ n\ x_n$.

Example:

- $x_0 := 0.5$

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- Want: $\{0.5 :: 0 :: 0 :: 0 :: \ldots, 0.5 :: 1 :: 1 :: 1 :: \ldots\}$

Nondeterministic dependent choice

Definition (Nondeterministic dependent choice)

Given a sequence $R : \mathbb{N} \to A \to A \to \mathrm{Prop}$, $x_0 : A$ and $f : \Pi(n : \mathbb{N}).\ \Pi(x_n : A).\ \mathrm{M}\Sigma(x_{n+1} : A).\ R\ n\ x_n\ x_{n+1}$.
There is $F : \mathrm{M}(\mathbb{N} \to A)$ such that for any $f \in F$ and $n \in \mathbb{N}$, $R\ n\ (f\ n)\ (f\ (n+1))$ holds.

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- $x_0 := 0.5$
- $f\ n\,x := \begin{cases} 0 \text{ or } 1 & \text{if } n = 0, \\ x & \text{otherwise} \end{cases}$
- $R\ \ n\ \ x\ \ y := x > 0 \to x = y$

Nondeterministic limits

Brauße and Müller suggested the following nondeterministic limit based on refinements. To construct a multivalued limit $X : \mathrm{M}R$,

- Provide a 2^{-0} approximation to some limit $x \in X$
- Provide a nondeterministic refinement function $f : \mathbb{N} \to R \to \mathrm{M}R$
- Whenever x_n is a 2^{-n} approx. to some limit $x \in X$, any $x_{n+1} \in f\ n\ x_n$ is a $2^{-(n+1}$ approximation to some (not necessarily the same) $x \in X$ and $|x_n - x_{n+1}| \leq 2^{-(n+1)}$

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This limit is derivable from the nondeterministic dependent choice: Choose $R\ n\ x\ y := |x - y| \leq 2^{-(n+1)} \wedge (y \sim_n X)$

Nondeterministic limits with additional information

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- For example, for the square root example we choose $Q\ n\ x := (|z| \leq 2^{-(n+2)} \wedge x = 0) + (x \cdot x = z)$.
- We not only need to refine the approximation but also construct a Boolean term in each step.

Program Extraction

When we prove a statement

$$\Pi(x : R).\ \Sigma(y : R).\ P\ x\ y$$

we get a computable function $f : \mathbb{R} \to \mathbb{R}$ computing numbers with the property P.

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In the Coq implentation this is realized by mapping R to AERN's datatype `CReal` and axiomatized operations to primitive operations in AERN.

Quality of programs: Reliability

Need to trust:

- Coq core
- Coq extraction
- Haskell compiler, base libraries
- AERN

Quality of programs: Smooth development

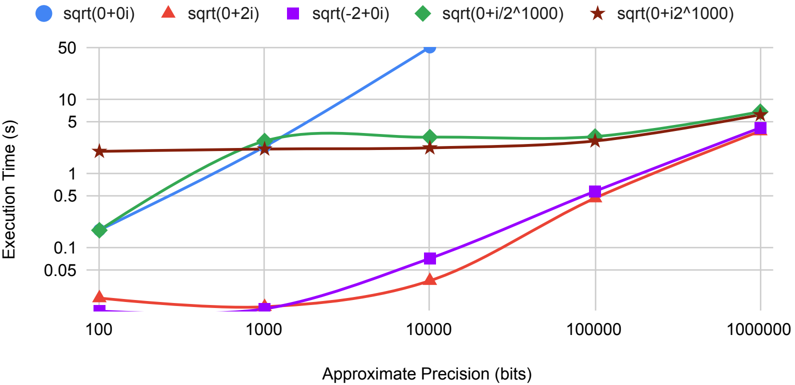
- Specifications readable
- Coq libraries can be used
- Algorithms readability still an issue

Quality of programs: Execution speed

Benchmark		Average execution time (s)			
Formula	Accuracy	Extracted	Hand-written	Native	iRRAM
$\max(0, \pi - \pi)$	10^6 bits	3.5	3.8	3.8	1.59
$\sqrt{2}$	10^6 bits	0.72	0.70	0.40	0.62
$\sqrt{\sqrt{2}}$	10^6 bits	1.52	1.38	0.85	1.15
$x - 0.5 = 0$	10^3 bits	1.44	0.32	—	0.03
$x(2-x) - 0.5 = 0$	10^3 bits	2.02	0.35	—	0.04
$\sqrt{x + 0.5} - 1 = 0$	10^3 bits	12.9	2.35	—	0.29

(i7-4710MQ CPU, 16GB RAM, Ubuntu 18.04, Haskell Stackage LTS 17.2)

Quality of programs: Execution speed



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Summary and Future Work

Summary:

- Real numbers are a primitive data-type in exact real computation frameworks
- We defined axiomatic reals in a dependent type theory
- We implemented the axioms in Coq and adjusted the extraction mechanism such that axiomatic reals are mapped to AERN's primitive data-type `CReal`, getting efficient and certified real number computation programs.

Thank you!