Computational Properties of Differential Equations Solution Operators

Svetlana Selivanova

KAIST

Based on joint works with: Ivan Koswara (KAIST), Gleb Pogudin (École Polytechnique), Victor Selivanov (A.P.Ershov Institute of Informatics Systems), Florian Steinberg (Inria), Holger Thies (Kyoto University), Martin Ziegler (KAIST)

The Second Korea Logic Day

January 14, 2022

▷ **Introduction and Motivation**

We consider differential equations (ODEs, PDEs)

$$\begin{cases} \frac{\partial}{\partial t}\vec{u} \;=\; \mathcal{A}\vec{u}, \\ \vec{u}(0) = \vec{\varphi} \end{cases}$$

$\triangleright$ **Introduction and Motivation**

We consider differential equations (ODEs, PDEs)

$$\begin{cases} \frac{\partial}{\partial t}\vec{u} \; = \; \mathcal{A}\vec{u}, \\ \vec{u}(0) = \vec{\varphi} \end{cases}$$

- ODE: $\mathcal{A}\vec{u} = f(t, \vec{u})$; one variable $\vec{u} = \vec{u}(t)$

- PDE: $\mathcal{A}\vec{u} = L\Big(t, \vec{x}, \vec{u}, \frac{\partial \vec{u}}{\partial t}, \frac{\partial \vec{u}}{\partial x_i}, \ldots\Big)$; several variables $\vec{u} = \vec{u}(t, \vec{x})$

Include important equations of mathematical physics. We consider initial-value and boundary-value problems

Introduction and Motivation

$$\begin{cases} \frac{\partial}{\partial t}\vec{u} \; = \; \mathcal{A}\vec{u}, \\ \vec{u}(0) = \vec{\varphi} \end{cases}$$

We study computational complexity of *solution operators*

$$\mathcal{A}, \varphi \to u.$$

- Is there an algorithm which computes the solution u from φ (with fixed coefficients of $\mathcal{A}$)? In which classes of functions? $||\varphi - \varphi^j|| < 2^{-j} \mapsto ||\mathbf{u} - u^n|| < 2^{-n}$

- From the coefficients of $\mathcal{A}$?

- What is the complexity of computations w.r.t. n (the bit length of the input) ? EXP: $O(2^n)$, P: $O(n^k)$, $O(log(n^k))$ ...?

- What is the optimal complexity bound? Which algorithm provides it?

▷ Introduction and Motivation

$$\begin{cases} \frac{\partial}{\partial t}\vec{u} \; = \; \mathcal{A}\vec{u}, \\ \vec{u}(0) = \vec{\varphi} \end{cases}$$

We study computational complexity of *solution operators*

$$\mathcal{A}, \varphi \to u.$$

Motivation:

- Uniform framework of computation, in particular rigorous foundation of numerical methods.
- Exact Real Computation algorithms for solving with guaranteed precision 2^{-n} and no floating point errors: important to perform bit-cost analysis (w.r.t. n) and devise optimal reliable algorithms. Sewon Park, Franz Brauße, Pieter Collins, SunYoung Kim, Michal Konečný, Gyesik Lee, Norbert Müller, Eike Neumann, Norbert Preining, Martin Ziegler. *Foundation of Computer (Algebra) ANALYSIS Systems: Semantics, Logic, Programming, Verification (arXiv) 2021*
- Particularly important for safety critical and small scale applications.

$\triangleright$ Main approaches

I. ("Discrete") Bit-complexity of *solution operators* $\mathcal{A}, \varphi \to u$ depending on the field where the coefficients of $\mathcal{A}$ belong to.

- $\mathbb{A}$ and its extensions

- PR subfields of $\mathbb{R}_c$

II. ("Exact Real") Bit-complexity.

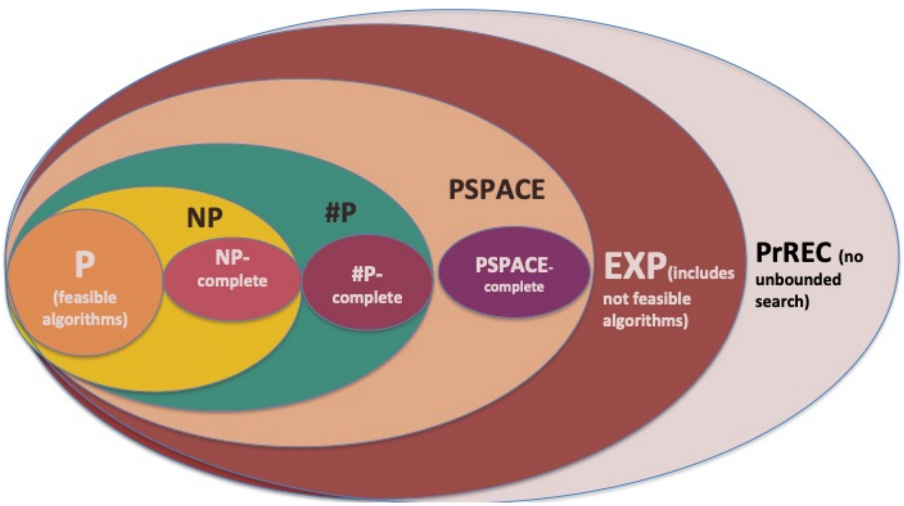
1. Pointwise computation of *solutions* $u(t)$ or $u(t, x)$ from given $\mathcal{A}$, φ with certain smoothness properties
   - C^∞ versus Analytic
   - C^k depending on k: which computability properties might improve and why

2. Parametrized complexity of *solution operators* $\mathcal{A}, \varphi \to u$

▷ Main approaches

I. ("Discrete") Bit-complexity of *solution operators* $\mathcal{A}, \varphi \to u$ depending on the field where the coefficients of $\mathcal{A}$ belong to.

II. ("Exact Real") Bit-complexity.



$$\mathbb{R}NC^i \subset \mathbb{R}PTIMESPACE(\log^i n) \subseteq$$
$$\subseteq \mathbb{R}NC^{2i} \ldots \subseteq \mathbb{R}PTIME \subseteq \mathbb{R}\#P_1 \subset$$
$$\subseteq \mathbb{R}\#P \subseteq \mathbb{R}\#P^{\#P} \subseteq$$
$$\subseteq \mathbb{R}PSPACE = \mathbb{R}PAR \subseteq \mathbb{R}EXP$$

$\triangleright$ **Examples**

ODEs: computability+complexity $\begin{cases} \frac{d}{dt}\vec{u} = f(t, \vec{u}), \\ \vec{u}(0) = \vec{v} \end{cases}$

• f polynomial or analytic $\Longrightarrow$ P (Bournez, Graça, Pouly 2011)

• f Lipschitz or C^1 $\Longrightarrow$ PSPACE complete (Kawamura 2010, Kawamura, Ota, Rösnick, Ziegler 2014)

• f continuous $\Longrightarrow$ noncomputable (Pour El, Richards 1982); "computable" on ordinal TMs (Bournez, Ouazzani 2019)

▷ **Examples**

PDEs computability

- Initial-value problems for the $3D$ wave equation

$$\begin{cases} u_{tt} = \Delta u \text{ on } [0, \infty) \times \mathbb{R}^3, \\ u\mid_{t=0} = f, u_t \mid_{t=0} = 0 \end{cases}$$

  (Weihrauch, Zhong 2002; Pour El, Richards 1989)

- Korteveg de Vries equation $u_t + u_{xxx} - 6uu_x = 0$ (Gay, Zhang, Zhong 2001; Weihrauch, Zhong 2005)

- Linear and nonlinear Schrödinger equations;

- $Pu = \sum_{|\alpha| \le M} c_\alpha D^\alpha u = f$ (Weihrauch, Zhong 2006)

- Navier-Stokes equations (Ming Sun, Zhong, Ziegler 2015)

$\triangleright$ **Examples**

PDEs complexity

$\diamond$ Dirichlet problem for the Laplace equation

$$\Delta u = f \text{ on } B_d(0,1);$$

$$u = 0 \text{ on } \partial B_d(0,1)$$

(1) "in $\sharp\mathbf{P}$", (2) "$\sharp\mathbf{P}_1$-hard" [Kawamura, Steinberg, Ziegler 2017].

(here $\Delta u = \sum_{j=1}^{d} \frac{\partial^2}{\partial x_j^2} u$)

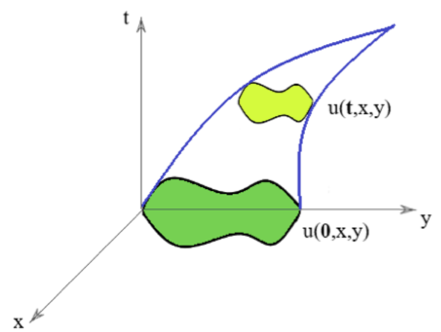
PART I

- ▷ Symmetric Hyperbolic Systems and complexity questions about their solution operators
- ▷ Bit-complexity for the algebraic real case: when does EXP turn into PTIME?
- ▷ Primitive recursiveness of the solution operator

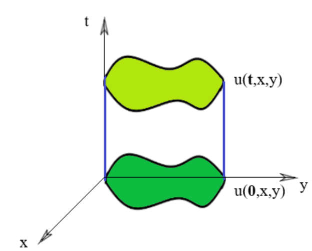
▷ **Symmetric Hyperbolic systems** (Friedrichs 1954)

We consider **initial-value** (IVP) and **boundary-value problems** (BVP) for symmetric hyperbolic systems of PDEs ($\mathbf{A} = \mathbf{A}^* > \mathbf{0}$, $\mathbf{B_i} = \mathbf{B_i^*}$)

$$(1) \begin{cases} A\frac{\partial}{\partial t}\vec{u} + \sum_{i=1}^{m} B_i \frac{\partial}{\partial x_i}\vec{u} = f(t,\vec{x}), \vec{x} \in \Omega, \\ \vec{u}(0,\vec{x}) = \varphi(\vec{x}), \end{cases}$$



$$(2) \begin{cases} A\frac{\partial}{\partial t}\vec{u} + \sum_{i=1}^{m} B_i \frac{\partial}{\partial x_i}\vec{u} = f(t,\vec{x}), \vec{x} \in \Omega, \\ \vec{u}(0,\vec{x}) = \varphi(\vec{x}), \\ \mathcal{L}\vec{u} \mid_{\partial\Omega} = 0 \end{cases}$$



▷ Symmetric Hyperbolic systems

Example: $2D$ (similarly 1D or 3D) acoustics equations (1st order system); also the wave equation $\frac{\partial^2}{\partial t^2}u - \Delta u = f$ reduces to such a system!

$$\begin{cases} \rho_0 \frac{\partial u}{\partial t} + \frac{\partial p}{\partial x} = 0, \\ \rho_0 \frac{\partial v}{\partial t} + \frac{\partial p}{\partial y} = 0, \\ \frac{\partial p}{\partial t} + \rho_0 c_0^2 (\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}) = 0, \end{cases}$$

where u, v are the velocities, p is the pressure, ρ_0 is the density and c_0 is the sound speed constant.

$$A = \begin{pmatrix} \rho & 0 & 0 \\ 0 & \rho & 0 \\ 0 & 0 & 1 \end{pmatrix}, B_1 = \begin{pmatrix} 0 & 0 & \rho c_0^2 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, B_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & \rho c_0^2 & 0 \end{pmatrix}$$

Also: elasticity, electromagnetism (Maxwell) equations etc.

▷ Symmetric Hyperbolic systems: complexity questions

We study: computability properties of the solution operator

$$R : (A, B_i, f, \varphi) \mapsto \mathbf{u}$$

Questions:

- Is the solution operator **computable** from f, φ? In which classes of functions? $||\varphi - \varphi^j|| < 2^{-j}$, $||f - f^j|| < 2^{-j} \mapsto ||\mathbf{u} - u^j|| < 2^{-n}$

- From the matrix coefficients A, B_i? $||A - A^j|| < 2^{-j}$, $||B_i - B_i^j|| < 2^{-j}$, $||\varphi - \varphi^j|| < 2^{-j}$, $||f - f^j|| < 2^{-j} \mapsto ||\mathbf{u} - u^n|| < 2^{-n}$

- What is the **complexity** of computations? (and which is optimal?) EXPTIME=$O(2^{\mathbf{n}})$, PTIME=$O(\mathbf{n}^k)$, LOGTIME=$O(log(\mathbf{n}))$ ...? (n is the bit length of the input)

- Does the solution operator preserve PTIME? And PrREC?

Results (joint work with V. Selivanov) about computability (2017) and complexity (2018,2021)

I. The solution operator

$$(\varphi, f) \mapsto \mathbf{u}$$

of (1), (2) is computable provided that the first and second partial derivatives of φ, f are uniformly bounded. Input: in sup-norm, output: in L_2-norm.

II. 1) The operator of the domain of existence and uniqueness $(A, B_1, \ldots, B_m) \mapsto H$ is computable (H is an intersection of $t \geq 0$, $x_i - \lambda_{\max}^{(i)} t \geq 0$, $x_i - 1 - \lambda_{\min}^{(i)} t \leq 0$, $(i = 1, \ldots, m)$, where $\{\lambda_k^{(i)}\}_{k=1}^n$ are the **eigenvalues** of $A^{-1}B_i$. Assume $\lambda_{\min}^{(i)} < 0 < \lambda_{\max}^{(i)}$ for all $i = 1, \ldots, m$.);

2) The solution operator

$$(\varphi, f, A, B_1, \ldots, B_m, n_A, n_1, \ldots, n_m) \mapsto \mathbf{u}$$

of (1), (2) is computable under certain additional spectral conditions on A, B_i.

Here n_A is the cardinality of spectrum of A (i.e. the number of different eigenvalues); n_i are the cardinalities of spectra of the matrix pencils $\lambda A - B_i$.

Eigenvectors are in general not computable! (Ziegler, Brattka)

3) The solution operator $(\varphi, f, A, B_1, \ldots, B_m) \mapsto \mathbf{u}$ of (1), (2) is computable when the coefficients of A, B_i run through an arbitrary computable real closed subfield of $\mathbb{R}$ (e.g. the set $\mathbb{A}$ of algebraic reals, or the real closure of $\mathbb{A} \cup \{c_1, \ldots, c_p\}$, $c_j \in \mathbb{R}_c$).

Comment: even though $\mathbb{R}_c$ is real closed, it is not constructivizable; for the finite subset $\{c_1, \ldots, c_p\}$, the corresponding real closure is even strongly constructivizable.

▷ **Bit-complexity for coefficients from $\mathbb{A}$: EXP or PTIME?**

III. We investigate **complexity bounds** for computing the solution operator of (1), (2) with **guaranteed precision**. From several approaches to measure the complexity of computation, we choose the classical computational complexity often referred to as **bit complexity**.

Informally: 1) from given $A, B_1, \ldots, B_m, \varphi, f$ and precision $\frac{1}{a}$ (where a is a positive integer), **find** approximation to H (domain of existence and uniqueness) and $\mathbf{u}$ (the precise solution of the Cauchy problem (1) or the boundary-value problem (2)).

2) Estimate the **computation time** needed to achieve the prescribed precision.

$\triangleright$ **Bit-complexity for coefficients from $\mathbb{A}$: EXP or PTIME?**

From various possible specifications of input data and parameters we stick to the following particular case:

$f = 0$;

A, B_i $(i = 1, 2, \ldots, m)$ are rational (or real algebraic) $n \times n$ matrices such that $A = A^* > 0$, $B_i = B_i^*$,

φ_j $(j = 1, 2, \ldots, n)$ are rational polynomials (or any class of functions with PTIME computable second derivatives).

The matrices and polynomials are encoded in a standard way by binary words; $\alpha \in \mathbb{A}$ is encoded via (p_α, k).

$\triangleright$ Bit-complexity for coefficients from $\mathbb{A}$: EXP or PTIME?

Let $m, n \geq 2$ be fixed positive integers. We search for an algorithm (and its complexity estimation) which, for any given integer $a > 1$, matrices $A, B_1 \ldots, B_m \in M_n(\mathbb{Q})$ or $M_n(\mathbb{A})$ and polynomials $\varphi_1 \ldots, \varphi_n \in \mathbb{Q}[x_1 \ldots, x_m]$, computes a rational $T > 0$ s.th. $H \subseteq Q \times [0, T]$, a spatial rational *grid step* h *dividing* 1, *a time grid step* τ *dividing* T *and a rational* h, τ*-grid function* $v : G \to \mathbb{Q}^n$ such that

$$||\mathbf{u} - \tilde{v}|_H||_{\mathbf{sL_2}} < \frac{1}{a}$$

where $\tilde{v}$ is the multilinear interpolation of v.

Throrem. This problem for (1), (2) can be solved in **EXP**.

$\triangleright$ Bit-complexity for coefficients from $\mathbb{A}$: EXP or PTIME?

Theorem. **PTIME** if (for $A, B_i(\mathbb{A})$):

m, n, a, M are fixed positive integers;

the quantities $||A||_2$, $\frac{\lambda_{max}(A)}{\lambda_{min}(A)}$,

$$\max_i \Big\{ ||B_i||_2, ||(A^{-1}B_i)^2||_2, \max_k \{|\mu_k| : \det(\mu_k A - B_i) = 0\}, \sup_{t,x} ||\frac{\partial^2 f}{\partial x_i \partial t}(t,x)||_2 \Big\},$$

and

$$\max_{i,j} \Big\{ ||A^{-1}B_iA^{-1}B_j - A^{-1}B_jA^{-1}B_i||_2, \sup_{t,x} ||\frac{\partial^2 f}{\partial x_i \partial x_j}(t,x)||_2, \sup_x ||\frac{\partial^2 \varphi}{\partial x_i \partial x_j}(x)||_2 \Big\}$$

are bounded by M.

▷ **Bit-complexity for coefficients from $\mathbb{A}$: EXP or PTIME?**

The proof heavily relies on:

• - using a stable difference scheme approximating (1), (2) and results on its convergence;

-proofs of the existence and uniqueness theorems for (1);

-properties of multilinear interpolations.

• deep results of computer algebra for polynomial arithmetic and computations in the fields of algebraic reals due to Loos, Collins, Grigoriev etc. and those recently considered by Alaev and Selivanov (including PTIME-presentability of $\mathbb{A}$ and PTIME computability of root finding).

$\triangleright$ Bit-complexity for coefficients from $\mathbb{A}$: EXP or PTIME?

• Polynomial-time computability (in some fields of algebraic reals) of finding eigenvectors of matrix pencils $\lambda A - B_i$ (recall that this problem is not computable in the field of reals). In particular, this is crucial for finding in polynomial time steps h, τ guaranteeing the stability of the difference scheme.

• Our proof is a mix of methods typical for symbolic and numerical computations.

• Our methods apply only to algebraic matrices because it is currently open whether there is a PTIME-presentable real closed field of reals which contains a transcendental number.

▷ Primitive recursiveness of the solution operator

T h e o r e m. Let $M, p \geq 2$ be integers. Then the solution operator $(A, B_1, \ldots, B_m, \varphi) \mapsto \mathbf{u}$ for (1) is a PR-computable function (uniformly on m, n) from $S_+ \times S^m \times C_s^{p+1}(Q, \mathbb{R}^n)$ to $C^p_{sL_2}(H, \mathbb{R}^n)$ where S and S^+ are respectively the sets of all symmetric and symmetric positively definite matrices from $M_n(\widehat{\mathbb{A}})$, $||\frac{\partial \varphi}{\partial x_i}||_s \leq M$ and $||\frac{\partial^2 \varphi}{\partial x_i \partial x_j}||_s \leq M$ for $i, j = 1, 2, \ldots, m$.

Here $\widehat{\mathbb{A}}$ is primitively recursively Archimedian subfield of $\mathbb{R}$, with PR splitting

T h e o r e m. Let $M, p \geq 2$ be integers and $A, B_1, \ldots, B_m \in M_n(\mathbb{R}_p)$ be fixed matrices satisfying the conditions in (1). Then the solution operator $\varphi \mapsto \mathbf{u}$ for (1) is a PR-computable function (uniformly on m, n) from $C_s^{p+1}(Q, \mathbb{R}^n)$ to $C^p_{sL_2}(H, \mathbb{R}^n)$, with the same constraints on φ as in the previous theorem.

PART II

- $\triangleright$ Real complexity classes

- $\triangleright$ Real complexity of PDEs depending on smoothness of initial data

- $\triangleright$ Parametrized complexity of solution operators

▷ **Brief reminder about complexity concepts**

- Ker-I Ko. Complexity Theory of Real Functions, 1991.
- Klaus Weihrauch. Computable Analysis, 2000.

Real complexity classes

◇ For real functions

Def. Computing $f :\subseteq \mathbb{R} \to \mathbb{R}$ in time $t : \mathbb{N} \to \mathbb{N}$ means, on input $a_m \in \mathbb{Z}$ s.th.

$$|x - a_m/2^m| \le 1/2^m,$$

to output $b_n \in \mathbb{Z}$ s.th.

$$|f(x) - b_n/2^n| \le 1/2^n,$$

in $\le t(n)$ steps.

- **PTIME** if t(n)=poly(n)
- **EXP** if t(n)=exp(n)
- **PSPACE**: if the amount of memory s(n) is bounded polynomially in n

$\triangleright$ **Examples**

$\diamond$ The following are equivalent:

- FP=$\sharp P$

- For every polynomial time computable $h : [0,1] \to \mathbb{R}$, the function
$$x \to \int_0^x h(t)dt$$
is again polynomial time computable.

(In other words, indefinite Riemann integration is **"$\sharp P$-complete"**)

Reminder: $\sharp P = \{f : \mathbb{N} \to \mathbb{N} \mid f$ counts the number of accepting computations of a non-deterministic polynomial-time Turing machine$\}$

If h is analytic, then P-computable!

▷ Main result about bit-complexity

Theorem. (Koswara, Pogudin, S., Ziegler) Suppose the given IVP and BVP be well posed and admit a converging finite difference approximation (with certain natural properties).

$B_i(x)$, $\varphi(x)$ fixed PTIME computable functions. Then:

1. The solution u is in **PSPACE**

2. For the periodic boundary condition u is **"in $\sharp P$"**.

Examples to which this theorem applies:

1. Heat equation $\frac{\partial}{\partial t}u = \Delta u$ for $u(t, x, y) \in C^{(1,4)}([0, T] \times \bar{\Omega})$ (convergence w.r.t. sup norm!)

2. 2D Wave equation $\frac{\partial^2}{\partial t^2}u = \Delta u$ for $u(t, x, y) \in C^{(4,5)}([0, T] \times \bar{\Omega})$ (convergence w.r.t. sup norm!)

3. Symmetric hyperbolic systems (including acoustics, elasticity, Maxwell equations): smoothness conditions are those to guarantee convergence of the difference scheme in sup-norms.

Remark: for *computability*, C^2 was enough, but w.r.t. L_2 norm and with additional assumptions [Selivanov, S. 2017]

▷ **Theorems about real complexity of DEs**

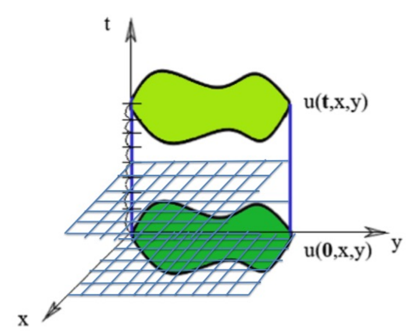
Some proof ideas

$$\frac{\partial}{\partial t}\vec{u} = \sum_{i=1}^{m} B_i(x)\frac{\partial}{\partial x_i}\vec{u}, \quad \vec{u}(0,x) = \varphi(x), \quad (\mathcal{L}\vec{u}\mid_{\partial\Omega} = 0).$$

Discretize with uniform grid steps τ, $h = 2^{-O(2^n)}$

$$\mathbf{u}^{(n)} = \mathbf{A_n}^{2^n}\varphi^{(n)}$$

Huge matrix powering!



Dimension of A_n is $O(2^n)$; powers are **uniformly bounded**

▷ **Theorems about real complexity of PDEs**

Lemmas

- 2^n vector $\times$ 2^n vector: $\#P$-complete
- 2^n matrix to the power 2^n: $PSPACE$-complete
- For the special case of periodic PDEs, 2^n matrix to the power 2^n is in $\#P$ (for 2-band matrices also in $PTIME$)

$\triangleright$ **Theorems about real complexity of DEs** (Koswara, S., Ziegler)

$$\frac{\partial}{\partial t}\vec{u} = \sum_{i=1}^{m} B_i(x)\frac{\partial}{\partial x_i}\vec{u}, \qquad \vec{u}(0) = \varphi(x).$$

Theorem. If φ, B_i are **analytic**, then: φ, $B_i \in$ PTIME $\Longrightarrow \mathbf{u} \in$ PTIME

$$\vec{u}(t) \;=\; \sum_{k=0}^{\infty} \frac{t^k}{k!}\Big(\sum\nolimits_{i=1}^{m} B_i(x)\frac{\partial}{\partial x_i}\Big)^k \varphi(x)$$

Theorem. Given $A \in [-1;1]^{d\times d}$ and $\vec{v} \in [-1;1]^d$, the solution to the first-order system of linear **ODEs**

$$\tfrac{\partial}{\partial t}\vec{u}(t) \;=\; A\vec{u}(t), \quad t \in [0;1], \quad \vec{u}(0) = \vec{\varphi}$$

is

$$\vec{u}(t) = \exp(tA)\vec{\varphi} := \sum_k \frac{t^k}{k!}A^k\vec{\varphi}$$

and computable in space $O\big(\log(n)\cdot(\log d + \log n)\big)$.

▷ **Parametrized complexity**

1) $$\begin{cases} \frac{\partial}{\partial t}\vec{u} = \sum_{i=1}^{m} B_i(\vec{x})\frac{\partial}{\partial x_i}\vec{u}, \vec{x} \in \Omega, \\ \vec{u}(0,\vec{x}) = \varphi(\vec{x}), \end{cases}$$

- φ, B_i are analytic; the power series coefficents are bounded:

$$|a_\alpha| \leq M \cdot L^{|\alpha|}, \qquad \text{for all } \alpha \in \mathbb{N}^d$$

- **Theorem.** Fix $d \in \mathbb{N}$ and consider the solution operator that maps any analytic right-hand sides $B_1, \ldots, B_d : [-1;1]^d \to \mathbb{C}^{d' \times d'}$ and initial condition $\varphi : [-1;1]^d \to \mathbb{C}^{d'}$ and 'small' enough $t \in \mathbb{C}$ to the solution $u = u(t, \cdot)$.
  This operator is computable in parameterized time polynomial in $n + L + \log M$, where $\varphi, B_1, \ldots, B_d$ are given via their (componentwise) Taylor expansions around $\vec{0}$ as well as integers (M, L) as coefficient bounds to $\varphi, B_1, \ldots, B_d$ componentwise.

▷ Summary on uniform results: computing solution operators

$$\frac{\partial}{\partial t}\vec{u} = \sum_{i=1}^{m} B_i(x)\frac{\partial}{\partial x_i}\vec{u}, \quad \vec{u}(0) = \varphi(x).$$

(S.,Steinberg, Thies, Ziegler) φ, B_i analytic PTIME computable

• The **operator** $(B_i, \vec{\varphi}) \mapsto \vec{u}$ is computable in parametrized time polynomial in $n + L + logM$, where M is the maximal coefficients bound for $B_i, \vec{\varphi}$; $1/L$ is the radius of convergence.

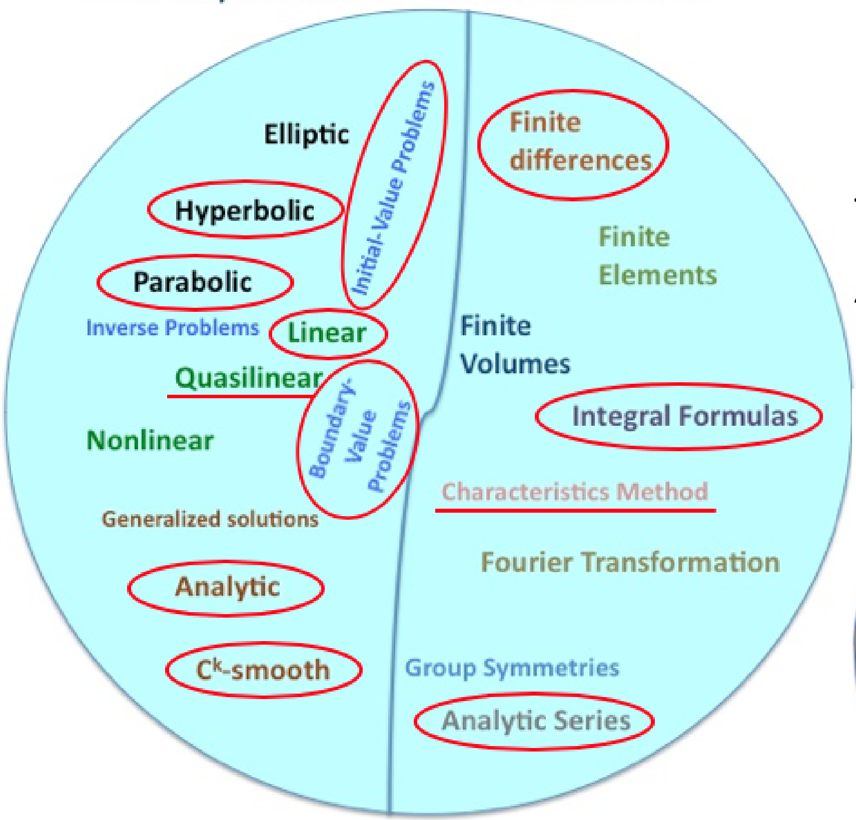
• For any $t > 0$ the solution of the Heat equation $u_t = \Delta u$ is computable in parametrized time polynomial in $n + \log t + L + \log M$.

(S., Selivanov)

• If $B_i = B_i^* \in M_n(\mathbb{R}_p)$, then the operator $\vec{\varphi} \mapsto \vec{u}$ is PR; uniformity on B_i for certain subfields.

• If $B_i = B_i^* \in M_n(\mathbb{A})$ and n is fixed, then the operator $\vec{\varphi} \mapsto \vec{u}$ is P uniformly on B_i for fixed output precision.

The variety of PDEs and methods to solve them



THANK YOU FOR YOUR ATTENTION!