Gödel's Incompleteness Theorem; sketch of the rigorous proof

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References

Byunghan Kim, ‘Complete proofs of Gödel's Incompleteness Theorems’, lecture note on Byunghan Kim's homepage:
https://web.yonsei.ac.kr/bkim/

Recursivevess of a function

Notation: $\omega = \mathbb{N}$, the set of natural numbers.

Definition

For $R \subseteq \omega^n$ a relation, $\chi_R : \omega^n \to \omega$, the characteristic function on R, is given by

$$\chi_R(\bar{a}) = \begin{cases} 1 & \text{if } \neg R(\bar{a}), \\ 0 & \text{if } R(\bar{a}). \end{cases}$$

Definition

A function from ω^n to ω ($n \geq 0$) is called **recursive** (or **computable**) if it is obtained by finitely many applications of the following 3 rules:

R1.
- $I_i^n : \omega^n \to \omega$, $1 \leq i \leq n$, defined by $I_i^n(x_1, \cdots, x_n) = x_i$ is recursive;
- $+ : \omega \times \omega \to \omega$ and $\cdot : \omega \times \omega \to \omega$ are recursive;
- $\chi_< : \omega \times \omega \to \omega$ is recursive.

Recursiveness of a function

Definition(Continued)

R2. (Composition) For recursive functions $G, H_1, \cdots, H_k$ such that $H_i : \omega^n \to \omega$ and $G : \omega^k \to \omega$, $F : \omega^n \to \omega$ defined by

$$F(\overline{a}) = G(H_1(\overline{a}), \cdots, H_k(\overline{a}))$$

is recursive.

R3. (Minimization) Let $G : \omega^{n+1} \to \omega$ be recursive, such that for all $\overline{a} \in \omega^n$ there exists some $x \in \omega$ such that $G(\overline{a}, x) = 0$. Then $F : \omega^n \to \omega$, defined by

$$F(\overline{a}) = \mu x(G(\overline{a}, x) = 0)$$

is recursive (where $\mu x P(x) = \min\{x \in \omega \mid P(x)\}$).

Recursivess of a relation

Definition

$R(\subseteq \omega^n)$ is called **recursive**, or **computable** (R is a recursive relation) if χ_R is a recursive function.

Digression: Church's Thesis

Let Ob be some countable set of 'objects'.

Definition

- A countable set $S \subseteq Ob$ is called **computable*** if there is an 'algorithm' determining the membership of S.
- A function $f : \omega^n \to \omega$ is **computable*** if there is an algorithm which 'effectively produces' $f(\overline{a})$ for given $\overline{a} \in \omega^n$.

Church's Thesis says that a function/relation is recursive(or Turing computable) if and only if it is computable*.

Coding of a sequence of numbers: β-function Lemma

Lemma (β-function Lemma)

There is a recursive function $\beta : \omega^2 \to \omega$ such that $\beta(a, i) \leq a \dot{-} 1$ for all $a, i \in \omega$, and for any $a_0, \cdots, a_{n-1} \in \omega$, there is $a \in \omega$ such that $\beta(a, i) = a_i$ for all $i < n$.

Definition

The **sequence number** of a sequence of natural numbers $a_1, \cdots, a_n$ is given by

$$< a_1, \cdots, a_n >= \mu x \Big((\beta(x, 0) = n) \wedge (\beta(x, 1) = a_1) \wedge \cdots \wedge (\beta(x, n) = a_n) \Big).$$

Remark. $<>$ is recursive and injective.

Representability Theorem

Let $\mathcal{L}_{\mathcal{N}} = \{+, \cdot, S, <, 0\}$. Q is an (finite) $\mathcal{L}_{\mathcal{N}}$-theory consists of

Q1. $\forall x(Sx \neq 0)$

Q2. $\forall x \forall y(Sx = Sy \to x = y)$

Q3. $\forall x(x + 0 = x)$

Q4. $\forall x \forall y(x + Sy = S(x + y))$

Q5. $\forall x(x \cdot 0 = 0)$

Q6. $\forall x \forall y(x \cdot Sy = x \cdot y + x)$

Q7. $\forall x(\neg x < 0)$

Q8. $\forall x \forall y(x < Sy \leftrightarrow x < y \lor x = y)$

Q9. $\forall x \forall y(x < y \lor x = y \lor y < x)$

Peano Arithmetic, or PA, is Q union generalizations of the following formulas:

$$(\varphi^x_0 \land \forall x(\varphi \to \varphi^x_{Sx})) \to \varphi.$$

Representability Theorem

Notation. $\underline{0} \equiv 0$, $\underline{1} \equiv S0$, $\underline{2} \equiv SS0, \cdots$, which are $\mathcal{L}_{\mathcal{N}}$-terms.

Theorem (Representability Theorem)

Every recursive function or relation is representable in Q. i.e. for recursive $f : \omega^n \to \omega$, there exists an $\mathcal{L}_{\mathcal{N}}$-formula $\varphi(x_1, \cdots, x_n, y)$ such that for all $k_1, \cdots, k_n \in \omega$,

$$Q \vdash \forall y \Big(\varphi(\underline{k_1}, \cdots, \underline{k_n}, y) \leftrightarrow y = \underline{f(k_1, \cdots, k_n)} \Big)$$

and recursive $P \subseteq \omega^n$, there exists an $\mathcal{L}_{\mathcal{N}}$-formula $\varphi(x_1, \cdots, x_n)$ such that for all $k_1, \cdots, k_n \in \omega$,

$$P(k_1, \cdots, k_n) \text{ implies } Q \vdash \varphi(\underline{k_1}, \cdots, \underline{k_n}) \text{ and}$$

$$\neg P(k_1, \cdots, k_n) \text{ implies } Q \vdash \neg\varphi(\underline{k_1}, \cdots, \underline{k_n}).$$

Coding of symbols

Let $\mathcal{L}$ be a countable language with $\mathcal{L} = \mathcal{C} \cup \mathcal{F} \cup \mathcal{P}$ and $\mathcal{V}$ a set of variables. $\mathcal{L}$ is called **reasonable** if the following two functions exist:

- $h : \mathcal{L} \cup \mathcal{V} \cup \{\neg, \rightarrow, \forall\} \rightarrow \omega$ injective such that each of $h(\mathcal{C}), h(\mathcal{F}), h(\mathcal{P}), h(\mathcal{V})$ is recursive.
- AR: $\omega \rightarrow \omega \setminus \{0\}$ recursive such that
  - $\text{AR}(h(f)) = n$, for n-ary function symbol f and
  - $\text{AR}(h(P)) = n$, for n-ary predicate symbol P.

Coding of formulas: Gödel numbers

$\ulcorner\urcorner : \{\mathcal{L}\text{-terms and } \mathcal{L}\text{-formulas}\} \to \omega$ inductively, by

- For $x \in \mathcal{V} \cup \mathcal{C}$, $\ulcorner x \urcorner = < h(x) >$.
- For $\mathcal{L}$-terms $t_1, \cdots, t_n$ and n-ary $f \in \mathcal{F}$,
$$\ulcorner f t_1 \cdots t_n \urcorner = < h(f), \ulcorner t_1 \urcorner, \cdots, \ulcorner t_n \urcorner > .$$
- For $\mathcal{L}$-terms $t_1, \cdots, t_n$ and n-ary $P \in \mathcal{P}$,
$$\ulcorner P t_1 \cdots t_n \urcorner = < h(P), \ulcorner t_1 \urcorner, \cdots, \ulcorner t_n \urcorner > .$$
- For $\mathcal{L}$-formulas φ and ψ,
$$(\ulcorner \to \varphi\psi \urcorner =)\ulcorner \varphi \to \psi \urcorner = < h(\to), \ulcorner \varphi \urcorner, \ulcorner \psi \urcorner >$$
$$\ulcorner \neg\varphi \urcorner = < h(\neg), \ulcorner \varphi \urcorner >$$
$$\ulcorner \forall x \varphi \urcorner = < h(\forall), \ulcorner x \urcorner, \ulcorner \varphi \urcorner > .$$

Remark. Gödel numbering $\ulcorner\urcorner$ is recursive and injective.

Axiomatizable and Decidable Theories

Definition

Let T be an $\mathcal{L}$-theory.

1. $\underline{T} = \{\ulcorner\sigma\urcorner \mid \sigma \in T\}$.
2. $\mathrm{Cn}\,T = \{\sigma \in \mathrm{Sent}(\mathcal{L}) \mid T \vdash \sigma\}$.
3. T is called **complete** if $\mathrm{Cn}\,T$ is maximal consistent.
   i.e. it is consistent and for any $\sigma \in \mathrm{Sent}(\mathcal{L})$, $\sigma \in \mathrm{Cn}\,T$ or $\neg\sigma \in \mathrm{Cn}\,T$.
4. T is **axiomatizable** if there exists a theory S such that $\mathrm{Cn}\,S = \mathrm{Cn}\,T$, such that $\underline{S}$ is recursive.
5. T is **decidable** if $\underline{\mathrm{Cn}\,T}$ is recursive.

Theorem

If T is axiomatizable and complete in $\mathcal{L}$, then T is decidable.

Technical Lemma and a Fact

Theorem (Gödel, Fixed Point Theorem)

For any $\mathcal{L}_{\mathcal{N}}$-formula $\varphi(x)$, there is some $\mathcal{L}_{\mathcal{N}}$-sentence σ such that

$$Q \vdash \sigma \leftrightarrow \varphi(\underline{\ulcorner \sigma \urcorner}).$$

With Representability Theorem, Fixed Point Theorem will play a critical role to derive a contradiction in the proof of a theorem in the next slide.

Fact (A result of Craig's Theorem)

PA is axiomatizable.

In particular, letting $T = PA$, $T \cup Q$ is consistent and by above Fact, T is axiomatizable.

Results: Strong Undecidability of Q

From now on, let $\mathcal{L}$ ($\supseteq \mathcal{L}_{\mathcal{N}}$) be countable reasonable and T be an $\mathcal{L}$-theory.

Theorem (Strong Undecidability of Q)

If $T \cup Q$ is consistent, then T is not decidable (i.e. $\underline{\mathrm{Cn}\ T}$ is not recursive).

Sketch of the Proof.

Suppose not, that is, $\underline{\mathrm{Cn}\ T}$ is recursive.
Then it can be shown that $\underline{\mathrm{Cn}(T \cup Q)}$ is recursive since Q is finite.
By Representability Theorem, there is $\varphi(x)$ representing $\underline{\mathrm{Cn}(T \cup Q)}$, i.e.

$$\text{for any } \tau,\ \text{if } \tau \in \mathrm{Cn}(T \cup Q),\ \text{then } Q \vdash \varphi(\underline{\ulcorner \tau \urcorner}) \text{ and}$$
$$\text{if } \tau \notin \mathrm{Cn}(T \cup Q),\ \text{then } Q \vdash \neg\varphi(\underline{\ulcorner \tau \urcorner}).$$

By Fixed Point Theorem, there is σ such that $Q \vdash \sigma \leftrightarrow \neg\varphi(\underline{\ulcorner \sigma \urcorner})$. Then

$$\begin{cases} \sigma \in \mathrm{Cn}(T \cup Q) \text{ implies } Q \vdash \neg\sigma \text{ and} \\ \sigma \notin \mathrm{Cn}(T \cup Q) \text{ implies } Q \vdash \sigma, \text{ hence contradiction for any case.} \end{cases}$$

Results: Gödel's Incompleteness Theorems

Now the First Incompleteness Theorem is a corollary of previous theorems.

Theorem (Gödel-Rosser, First Incompleteness Theorem)

If $T \cup Q$ is consistent and T is axiomatizable, then T is not complete.

Proof.

By previous theorems;
if T is axiomatizable and complete in $\mathcal{L}$, then T is decidable and
if $T \cup Q$ is consistent, then T is not decidable. □

Con_T is an $\mathcal{L}$-sentence that says '$0 \neq 0$ is not provable from T', which is equivalent to saying that 'T is consistent'.

Theorem (Gödel, Second Incompleteness Theorem)

If T is consistent, $\underline{T}$ is recursive and $T \vdash \mathrm{PA}$, then $T \nvdash Con_T$.