Exact Truthmaker Semantics in relation to multi-valued logic

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Outline

1. Basic ideas of truthmaker semantics
   - The Truthmaker principle
   - Formal framework
   - Exact truthmaker semantics for classical propositional logic
2. Exactification of multi-valued semantics
   - The idea of Exactification
   - Analysis of truth
   - Analysis of consequence

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The Truthmaker principle

- The Truthmaker Principle says that every true proposition is made true by something (Williamson 1999).
- That something is called a *state*.
- We do not attempt to further analyze what a state is; it is whatever does the job of truthmaking.
- For an intuitive example, consider the proposition P that the Empire State Building is between 33rd and 34th Streets in Manhattan.
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Exact and inexact verification

- A verifier for a proposition is said to be *exact* if it is entirely relevant to the truth of the proposition.
- And it is said to be *inexact* if it is partially relevant.
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Partial order

- One way of formalizing these basic ideas is given by Fine (2017) in terms of partial ordering.
- A *partial order* is an ordered pair $\langle \mathscr{S}, \sqsubseteq \rangle$, where $\sqsubseteq$ is a reflexive, transitive, and anti-symmetric binary relation on $\mathscr{S}$.
- A partial order $\langle \mathscr{S}, \sqsubseteq \rangle$ is said to be *complete* if and only if every subset S of $\mathscr{S}$ has a least upper bound ($\bigsqcup S$, in symbol).
- Notice that if a partial order is complete, there exist the $\sqsubseteq$-least and $\sqsubseteq$-greatest elements, called the *bottom* and *top* elements respectively.

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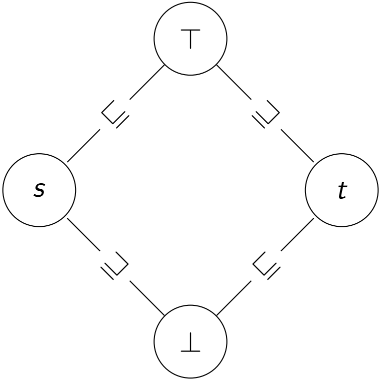
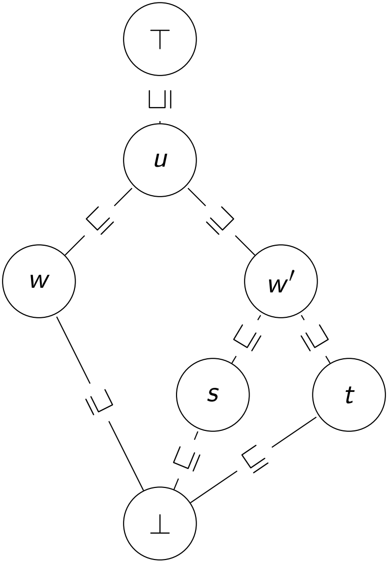


Example 2



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  - s: the presence of the Empire State Building in its location;
  - t: the presence of five pigeons on the top; and
  - $\top$: the presence of both.
- s and t are *part of* $\top$.
- So we would have: $s \sqsubseteq \top$, $t \sqsubseteq \top$, but neither $\top \sqsubseteq s$ nor $\top \sqsubseteq t$.
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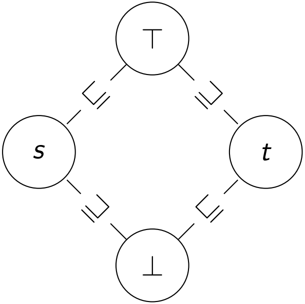

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Language

- Let P_1, P_2, ... be a countable list of propositional variables; we shall often use P, Q, R as metavariables for propositional variables.
- The well-formed formulas are constructed in the usual way, using the connectives $\neg$, $\wedge$, $\vee$.
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Truthmaker model

- Let Γ be a set of formulas.
- $At(\Gamma) = \{P : P$ occurs in some formulas of $\Gamma\}$.
- $Fml(\Gamma) =$ the formulas whose atomic subformulas are all in $At(\Gamma)$.
- A *truthmaker model* $\mathfrak{A}$ of Γ is an ordered triple $\langle \mathscr{S}, \sqsubseteq, v\rangle$, where
  1. $\langle \mathscr{S}, \sqsubseteq\rangle$ is a state space, and
  2. v is a valuation that takes each state $s \in \mathscr{S}$ to a pair $\langle [s]^+, [s]^- \rangle$ of subsets of $At(\Gamma)$.
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- A *truthmaker model* $\mathfrak{A}$ of Γ is an ordered triple $\langle \mathscr{S}, \sqsubseteq, v \rangle$, where
  1. $\langle \mathscr{S}, \sqsubseteq \rangle$ is a state space, and
  2. v is a valuation that takes each state $s \in \mathscr{S}$ to a pair $\langle [s]^+, [s]^- \rangle$ of subsets of $At(\Gamma)$.
- We require that for each $P \in At(\Gamma)$, there is at least one $s \in \mathscr{S}$ such that $P \in [s]^+ \cup [s]^-$.

Truthmaker model

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Exact verification/falsification

- Given a truthmaker model $\mathfrak{A}$ of Γ, the notions of exact verification and falsification (written $s \Vdash^+ A$ and $s \Vdash^- A$, respectively) can be defined as follows:

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\begin{array}{rcl}
\mathfrak{A}, s \Vdash^+ P & \Leftrightarrow & P \in [s]^+, \text{ for atomic } P; \\
\mathfrak{A}, s \Vdash^- P & \Leftrightarrow & P \in [s]^-, \text{ for atomic } P; \\
\mathfrak{A}, s \Vdash^+ \neg A & \Leftrightarrow & s \Vdash^- A; \\
\mathfrak{A}, s \Vdash^- \neg A & \Leftrightarrow & s \Vdash^+ A; \\
\mathfrak{A}, s \Vdash^+ A \wedge B & \Leftrightarrow & s = s_1 \sqcup s_2, \text{ for some } s_1, s_2 \\
& & \quad \text{with } s_1 \Vdash^+ A \text{ and } s_2 \Vdash^+ B; \\
\mathfrak{A}, s \Vdash^- A \wedge B & \Leftrightarrow & s \Vdash^- A \text{ or } s \Vdash^- B; \\
\mathfrak{A}, s \Vdash^+ A \vee B & \Leftrightarrow & s \Vdash^+ A \text{ or } s \Vdash^+ B; \\
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Exact verification/falsification

- Given a truthmaker model $\mathfrak{A}$ of Γ, and a formula $A \in Fml(\Gamma)$, let:

$$|A|^+ = \{s \in \mathscr{S} : s \Vdash^+ A\};$$
$$|A|^- = \{s \in \mathscr{S} : s \Vdash^- A\}.$$

- Given any $S, T \subseteq \mathscr{S}$, let's write

$$S \sqcup T = \{s \sqcup t : s \in S \text{ and } t \in T\};$$

Notice that $S \sqcup T = \emptyset$ if either $S = \emptyset$ or $T = \emptyset$.

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Exact verification/falsification

- With these notations, we can rewrite the inductive clauses as follows:

$$
\begin{aligned}
s \Vdash^{+} P &\quad\Leftrightarrow\quad s \in |P|^{+}, \text{ for atomic } P;\\
s \Vdash^{-} P &\quad\Leftrightarrow\quad s \in |P|^{-}, \text{ for atomic } P;\\
s \Vdash^{+} \neg A &\quad\Leftrightarrow\quad s \in |A|^{-};\\
s \Vdash^{-} \neg A &\quad\Leftrightarrow\quad s \in |A|^{+};\\
s \Vdash^{+} A \wedge B &\quad\Leftrightarrow\quad s \in |A|^{+} \sqcup |B|^{+}\\
s \Vdash^{-} A \wedge B &\quad\Leftrightarrow\quad s \in |A|^{-} \text{ or } s \in |B|^{-};\\
s \Vdash^{+} A \vee B &\quad\Leftrightarrow\quad s \in |A|^{+} \text{ or } s \in |B|^{+};\\
s \Vdash^{-} A \vee B &\quad\Leftrightarrow\quad s \in |A|^{-} \sqcup |B|^{-}
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Inexact verification and falsification

- A state s is an *inexact verifier* for A, written $s \triangleright^+ A$, if and only if s extends an exact verifier for A, i.e., $s' \sqsubseteq s$ for some $s' \Vdash^+ A$.
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Outline

Exactification

- *Exactification* is the idea due to Kit Fine that an inexact verifier (falsifier) for a proposition has an underlying exact verifier (falsifier).
- Consider a classical Boolean valuation, for example.
- It can be considered as a state in a model that verifies those formulas that are true—falsifies those formulas that are false—under the valuation.
- The relevant notions of verification and falsification are inexact.
  - The Boolean valuation determines the truth-value of every formula.
  - For any formula A, therefore, the Boolean valuation—conceived as a state—may have parts that are irrelevant to the truth or falsity of A.
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Auxiliary notions

- Let $\mathfrak{A} = \langle \mathscr{S}, \sqsubseteq, v \rangle$ be a model and s be a state in $\mathscr{S}$.
- s is said to be *atomically consistent* just in case there is no propositional variable P such that $s \rhd^{+} P$ and $s \rhd^{-} P$.
- s is said to be *atomically complete* just in case for all propositional variables P, $s \rhd^{+} P$ or $s \rhd^{-} P$.
- One notable feature of the current semantics is that it allows models to have states that are atomically inconsistent and/or incomplete.

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Outline

Correspondence with Belnapian valuations

- Due to the feature just mentioned, the current semantics naturally corresponds to Belnap's four-valued semantics, where each formula is assigned one of the four truth-values: True, False, Both, Neither.
- For each state s in $\mathfrak{A} = \langle \mathscr{S}, \sqsubseteq, v \rangle$ for Γ, define the corresponding four-valued assignment φ_s for propositional variables in $At(\Gamma)$:

$$\varphi_s(P) = \begin{cases} \{T\} & \text{if } s \rhd^+ P \text{ and } s \not\rhd^- P; \\ \{F\} & \text{if } s \not\rhd^+ P \text{ and } s \rhd^- P; \\ \{T, F\} & \text{if } s \rhd^+ P \text{ and } s \rhd^- P; \\ \emptyset & \text{if } s \not\rhd^+ P \text{ and } s \not\rhd^- P. \end{cases}$$

- A *Belnapian* valuation $\overline{\varphi_s}$ is a valuation extending φ_s to all formulas according to the scheme as given by Belnap.
- Obviously, then, for all formulas $A \in Fml(\Gamma)$, $s \rhd^+ A$ if and only if $T \in \overline{\varphi_s}(A)$, and $s \rhd^- A$ if and only if $F \in \overline{\varphi_s}(A)$.

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Analysis of "Truth under a Boolean valuation"

- Now, Boolean valuations can be considered as Belnapian valuations that assign only $\{T\}$ and $\{F\}$ to formulas.
- It is easy to see that those Belnapian valuations correspond to atomically consistent and complete states in the current semantics.
- This gives a simple truthmaker semantical analysis of the notion of "truth under a Boolean valuation."
- A formula is true (false) under a Boolean valuation just in case it has an exact verifier (falsifier) under an atomically consistent and complete state (in a model).
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Analysis of "Truth under a strong K3 valuation"

- In a similar fashion, Strong Kleene three-valued valuations can be considered as Belnapian valuations that assign $\{T\}$, $\{F\}$, $\emptyset$ to formulas.
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Analysis of "Truth under a LP valuation"

- Another well-known three-valued valuation is Priest's LP.
- LP valuations are Belnapian valuations that assign $\{T\}$, $\{F\}$, $\{T, F\}$ to formulas.
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Outline

Example: simple analysis of classical consequence

- Let us turn to the notion of consequence
- Consider as an example the following simple analysis of classical consequence.
- In the standard semantics, classical consequence is understood as the preservation of truth under all Boolean valuations.
- Truth (falsity) under a Boolean valuation is analyzed as inexact verification (falsification) under an atomically consistent and complete state.
- So, the most straightforward analysis of classical consequence would be in terms of the preservation of inexact verification under all modally sound and complete states (in all models).
  - A is a classical consequence of Γ if and only if in all models $\mathfrak{A} = \{\mathscr{S}, \sqsubseteq, v\}$ and for all modally sound and complete states in $\mathscr{S}$, $s \rhd^{+} A$ whenever $s \rhd^{+} B$ for all $B \in \Gamma$.

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- The set $\mathscr{D}$ of *designated* truth-values is any non-empty subset of the n-possible truth-values.
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Truthmaker semantic analysis of FDE-consequence

- Consider the Belnapian four-valued valuations.
- The FDE consequence relation is defined by by taking the set $\mathscr{D}$ of designated values to be $\{\{T\},\{T,F\}\}$.
  - A is an FDE-consequence of Γ just in case A is assigned a value in $\{\{T\},\{T,F\}\}$ under every Belnapian valuation that assigns to every formula in Γ a value in $\{\{T\},\{T,F\}\}$.
- Now recall that to every Belnapian valuation ϕ there corresponds a state s in a model such that, for any formula C,

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Truthmaker semantic analysis of LP-consequence

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Further generalization

- Now let us go back to the classical consequence relation.
- The problem is: how can we give an analysis of classical consequence without appeal to atomic completeness?
- A solution to this problem is to further generalize the notion of consequence by allowing two sets of designated values, one for Γ—the set of premises—and the other for A—the conclusion.
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Conclusion

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