Dialetheism on Arithmetic: Is There Any Inconsistent Primitive Recursive Relation?

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Contents

Claim. There is no inconsistent primitive recursive relation. (Because there is no primitive recursive function f in standard sence such that $f = f \neq f$ is true.)

- **Base 1.** If Priest's identity relation is inconsistent, then it is not primitive recursive in the standard sense.
- **Base 2.** If it is primitive recursive in the standard sense, then it does not have any primitive recursive function f such that it represents f and $f = f \neq f$.
- **Base 3.** Priest did not propose a new definition of 'primitive recursive function' that fits to his dialetheism.

What is paradox?

R.M. Sainsbury

'I understand by paradox: an apparently unacceptable conclusion derived by apparently acceptable reasoning from apparently acceptble premises.'

- **Question**: How can the acceptable premises lead by the acceptable reasoning to the unacceptable conclusion?
- **Three Solutions to the Paradoxes**
  (1) The premise is not actually acceptable.
  (2) The reasoning is not actually acceptable.
  (3) The conclusion is actually acceptable.

Dialetheism on Arithmetic

Dialetheism on Arithmetic:

1. Dialetheism is the view that there is a true contradiction.
2. In the finite models of arithmetic even numerical equations can be inconsistent; that is, there can be truths of the form $\mathbf{m} = \mathbf{n} \wedge \mathbf{m} \neq \mathbf{n}$.

History: Alfred Tarski (1944).The semantic conception of truth and the foundations of semantics.

If we now analyze the assumptions which lead to the [Liar paradox], we notice the following.

(I) We have implicitly assumed that the language in which the [paradox] is constructed contains, in addition to its expressions, also the names of these expressions, as well as semantic terms such as the term 'true' referring to sentences of this language; we have also assumed that all sentences which determine the adequate usage of this term can be asserted in the language. A language with these properties will be called 'semantically closed.'

History: Alfred Tarski (1944).The semantic conception of truth and the foundations of semantics.

A language $\mathfrak{L}$ is *semantically closed* if it is such that (i) for any formula φ in $\mathfrak{L}$, $\mathfrak{L}$ has a term $\ulcorner\varphi\urcorner$ which refers to φ, (ii) $\mathfrak{L}$ has a semantical concept in question, such as the term 'true,' and the term in $\mathfrak{L}$ satisfies its adequacy condition. For the example of the concept of truth, any instances of the concept satisfy T-schema.

History: Alfred Tarski (1944).The semantic conception of truth and the foundations of semantics.

Hierarchical T-schema: Let $\mathfrak{L}_{\mathbb{O}}$ be an object-language and $\mathfrak{L}_{\mathbb{M}}$ be a meta-language of $\mathfrak{L}_{\mathbb{O}}$. (i.e. any expressions in $\mathfrak{L}_{\mathbb{O}}$ is to be (translated) in $\mathfrak{L}_{\mathbb{M}}$ but not vice versa.) For any sentence φ in $\mathfrak{L}_{\mathbb{O}}$ and its name $\ulcorner\varphi\urcorner$ in $\mathfrak{L}_{\mathbb{M}}$,

$$\ulcorner\varphi\urcorner \text{ is true if and only if } \varphi.$$

History: Graham Priest (1986/2006). *In Contradiction: A Study of the Transconsistent*.

- The lesson of Gödel's proof shows that there is a true contradictrion even in arithmetic.
- Priest's finite inconsistent arithmetic is complete and inconsistent.
- It can use classical inferences except *ex contradictione quodlibet*.

History: Stewart Shapiro (2002). Incompleteness and Inconsistency.

Premise 1. If the *proof-representability relation** and the soundness of PA^* are acceptable, there is a primitive recursive relation φ such that $PA^* \vdash \varphi \wedge \neg\varphi$.

Premise 2. The *proof-representability relation** and the soundness of PA^* are acceptable.

Conclusion. There is a primitive recursive relation φ such that $PA^* \vdash \varphi \wedge \neg\varphi$.

History: Graham Priest (1986/2006). *In Contradiction: A Study of the Transconsistent*.

One has to accept that even primitive recursive relations may be inconsistent. But this is not news. In the finite models of arithmetic even numerical equations can be inconsistent; that is, there can be truths of the form $\boldsymbol{m} = \boldsymbol{n} \wedge \boldsymbol{m} \neq \boldsymbol{n}$. *Priest(2006, p. 260)*

The Gödelian argument for dialetheism.

Let *PA* be a standard first- or second-oder axiomatization of arithmetic. The language of *PA* contains the usual arithmetic terminology $\{0, s, +, \cdot\}$, but no semantic terminology. For each formula φ, let $\ulcorner\varphi\urcorner$ be the Gödel code of φ and, for some natural number n, $\overline{n}$ be its numeral. $\overline{\ulcorner\varphi\urcorner}$ is the numeral of the code of φ. Moreover, for some function f, we will use $\overline{f}$ for the relation that represents f. For any system S, '$S \vdash \varphi$' means that S derives φ and '$S \nvdash \varphi$' means that S does not.

The Gödelian argument for dialetheism.

Now augment the language with a new monadic truth predicate $T(x)$. The intended interpretation of $T(\ulcorner\varphi\urcorner)$ is that $\ulcorner\varphi\urcorner$ is the code of a true sentence in the *language of PA*. We introduce further terminology for satisfaction and augment *PA* with a standard Tarskian theory. We will then have, as theorems, each instance of the $T-$scheme:

$$T(\ulcorner\varphi\urcorner) \leftrightarrow \varphi. \tag{1}$$

Three options from the Gödelian argument for dialetheism

Now, we expand the induction principle of *PA* to apply to all formulas in the expanded language. That is, we apply induction to formulas that contain the semantic terminology. Call the resulting theory PA^*. The collection of PA^*-derivations is a recursive set of sequences of strings on the given alphabet. We can formulate a proof predicate $\overline{Prf_{PA^*}}(x,y)$ for PA^* that represents the primitive recursive function $Prf_{PA^*}(x,y)$ and says, 'x is a proof of y in PA^*.' $\overline{Prf_{PA^*}}(x,y)$ is a purely arithmetic formula even in the language of *PA*. That is to say, the relation $\overline{Prf_{PA^*}}$ represents PA^*-derivation. (Cf. Gödel(1931))

Three options from the Gödelian argument for dialetheism

The ***proof-representability relation****:

1. m is the code of a derivation in PA^* of the sentence whose code is n iff $\overline{Prf_{PA^*}}(\overline{m},\overline{n})$ is derivable in PA^*,
2. m is not the code of a derivation in PA^* of the sentence whose code is n iff $\neg\overline{Prf_{PA^*}}(\overline{m},\overline{n})$ is derivable in PA^*.

The **soundness of** PA^*:

$$PA^* \vdash \forall x(\exists y\overline{Prf_{PA^*}}(y,x) \to T(x)). \tag{2}$$

By the standard fixed-point construction, we have a Gödel sentence G^* such that

$$PA^* \vdash G^* \leftrightarrow \forall y\neg\overline{Prf_{PA^*}}(y,\overline{\ulcorner G^*\urcorner}) \tag{3}$$

is a theorem of PA^*. The excluded middle holds in PA^*, we have:

$$PA^* \vdash \exists y\overline{Prf_{PA^*}}(y,\overline{\ulcorner G^*\urcorner}) \vee \neg\exists y\overline{Prf_{PA^*}}(y,\overline{\ulcorner G^*\urcorner}). \tag{4}$$

Three options from the Gödelian argument for dialetheism

$$PA^* \vdash \exists y \overline{Prf_{PA^*}}(y, \overline{\ulcorner G^* \urcorner}) \vee \neg \exists y \overline{Prf_{PA^*}}(y, \overline{\ulcorner G^* \urcorner}).$$

In PA^*, we do a proof by cases. First, assume $\exists y \overline{Prf_{PA^*}}(y, \overline{\ulcorner G^* \urcorner})$. Then, from soundness, we have $T(\overline{\ulcorner G^* \urcorner})$, and do G^* by the relevant instance of the $T-$scheme. Now assume $\neg \exists y \overline{Prf_{PA^*}}(y, \overline{\ulcorner G^* \urcorner})$. But this is provably equivalent to G^*. So either way, G^* follows.

(a) By soundness and the right disjunct, we have $T(\ulcorner G^* \urcorner)$, and do G^* by the relevant instance of the $T-$scheme.
(b) The left disjunct is provably equivalent to G^*. So either way, G^* follows.

Three options from the Gödelian argument for dialetheism

Let g be the code of a derivation of G^* in PA^*. By the *proof-representability relation**, we have that

$$PA^* \vdash \overline{Prf_{PA^*}}(\overline{g^*}, \overline{\ulcorner G^* \urcorner}). \tag{5}$$

However, G^* is itself derivable in PA^*, and G^* is provably equivalent to

$$\forall y \neg \overline{Prf_{PA^*}}(y, \overline{\ulcorner G^* \urcorner}) \tag{6}$$

So we have that $\neg \overline{Prf_{PA^*}}(\overline{g^*}, \overline{\ulcorner G^* \urcorner})$ and thus

$$PA^* \vdash \overline{Prf_{PA^*}}(\overline{g^*}, \overline{\ulcorner G^* \urcorner}) \wedge \neg \overline{Prf_{PA^*}}(\overline{g^*}, \overline{\ulcorner G^* \urcorner}) \tag{7}$$

Three options from the Gödelian argument for dialetheism

While we consider that (1), (3), (4) are acceptable, we summarize the Gödelian argument for dialetheism in the following way:

Premise 1. If the *proof-representability relation** and the soundness of PA^* are acceptable, there is a primitive recursive relation φ such that $PA^* \vdash \varphi \wedge \neg\varphi$.

Premise 2. The *proof-representability relation** and the soundness of PA^* are acceptable.

Conclusion. There is a primitive recursive relation φ such that $PA^* \vdash \varphi \wedge \neg\varphi$.

Unlike the Gödelian argument for dialetheism, one may accept the following statement of the consistency of primitive recursive relations and reject that the *proof-representability relation** and the soundness of PA^* are acceptable.

The Consistency of Primitive Recursive Relations: For any arithmetic system S, there is no primitive recursive relation φ such that $S \vdash \varphi \wedge \neg\varphi$.

Three options from the Gödelian argument for dialetheism

In this regard, Shapiro(2002) offers three options to the argument

A. Accept the consistency of primitive recursive relations and the *proof-representability relation**, but the reject the soundness of PA^*.

B. Accept the consistency of primitive recursive relations and the soundness of PA^*, but reject *the proof-representability relation**.

C. Accept the *proof-representability relation** and the soundness of PA^*, but reject the the consistency of primitive recursive relations.

As the option C is for dialetheists, Priest(2006) accepted the option C as the simplest and most natural response. He rejected the option B because it involves jettisoning a connection in terms of which logicians have become accustomed to thinking.

Three options from the Gödelian argument for dialetheism

Interestingly, Priest rejected the option A by saying,

> *"... option A is obviously not the way to go. One has to accept that even primitive recursive relations may be inconsistent. But this is not news. In the finite models of arithmetic even numerical equations can be inconsistent; that is, there can be truths of the form* $\boldsymbol{m} = \boldsymbol{n} \wedge \boldsymbol{m} \neq \boldsymbol{n}$*".*

Priest's claim: there is an inconsistent primitive recursive relation.

The notion of ‘inconsistent primitive recursive relation

> *“... option A is obviously not the way to go. One has to accept that even primitive recursive relations may be inconsistent. But this is not news. In the finite models of arithmetic even numerical equations can be inconsistent; that is, there can be truths of the form* $\boldsymbol{m} = \boldsymbol{n} \wedge \boldsymbol{m} \neq \boldsymbol{n}$*.”*

Let $\approx$ be an identity relation expressing ‘x is identical with y.’ Priest’s argument has the following form:

Premise 1. $\approx$ is a primitive recursive relation.

Premise 2. There is an inconsistent model for arithmetic such that, for some natural numbers m and n, $\overline{m} \approx \overline{n} \wedge \overline{m} \not\approx \overline{n}$ is both true and false, and thus can be true.

Conclusion. There is an inconsistent primitive recursive relation.

The notion of ‘inconsistent primitive recursive relation

Premise 1. $\approx$ is a primitive recursive relation.

Premise 2. There is an inconsistent model for arithmetic such that, for some natural numbers m and n, $\overline{m} \approx \overline{n} \land \overline{m} \not\approx \overline{n}$ is both true and false, and thus can be true.

Conclusion. There is an inconsistent primitive recursive relation.

- Primitive recursive relations can be understood as any *n*-ary predicates that represent some primitive functions.

- The standard definition of ‘primitive recursiveness’ and ‘representability’ were introduced by Gödel(1931, 1934). What he called ‘recursive’ in is now called ‘primitive recursive.’ For instance, ‘*x* is identical with *y*’ and ‘*x* is a proof of *y*’ are primitive recursive relations.

The notion of ‘inconsistent primitive recursive relation

In logic, the expression ‘is consistent’ applies to a set of formulas or a system. ‘A relation is inconsistent’ is not often used.

There are two standard notions of consistency.

A syntactic consistency: for any system S, S is consistent if there is no φ such that S derives φ and $\neg\varphi$; otherwise it is inconsistent.

A semantic consistency: for any set S of formulas, S is consistent iff there is a case that every formula in S is true; otherwise it is inconsistent. Both notions apply a set of formulas (or a system), not a single relation.

The notion of ‘inconsistent primitive recursive relation

Definition

Let S be any system, f be any n-ary function, and φ be any relation. ‘$\varphi(n_1,\ldots,n_k,m)$ *represents* an n-ary function f in S’ means that for any natural number $n_1,\ldots,n_k$, $S \vdash \varphi(\overline{n_1},\ldots,\overline{n_k},\overline{m})$ if $f(n_1,\ldots,n_k) = m$. ‘$\varphi(n_1,\ldots,n_k,m)$ *represents* a predicate ψ in S’ means that for any natural number $n_1,\ldots,n_k$, $S \vdash \varphi(\overline{n_1},\ldots,\overline{n_k},\overline{m})$ if $\psi(n_1,\ldots,n_k)$ and $S \vdash \neg\varphi(\overline{n_1},\ldots,\overline{n_k},\overline{m})$ if $\neg\psi(n_1,\ldots,n_k)$. We say that φ is *primitive recursive* in S iff for some primitive recursive function f and predicate ψ, φ represents f (or ψ) in S.

Definition

Let φ be any relation. For some system S and some model $\mathbb{M}$ for S, φ is *inconsistent* in S relative to $\mathbb{M}$ iff $S \vdash \varphi \wedge \neg\varphi$ and $\varphi \wedge \neg\varphi$ is both true and false in $\mathbb{M}$.

Priest's identity as the inconsistent primitive recursive relation

Premise 1. $\approx$ is a primitive recursive relation.

Premise 2. There is an inconsistent model for arithmetic such that, for some natural numbers m and n, $\overline{m} \approx \overline{n} \wedge \overline{m} \not\approx \overline{n}$ is both true and false, and thus can be true.

Conclusion. There is an inconsistent primitive recursive relation.

Question 1. Does Priest's identity relation represent identity function?

Question 2. If not, does Priest's identity relation represent a primitive recursive function?

The identity as a congruence relation

The language $\mathscr{L}$ of *LP* is that of classical first-order logic, including function symbols and identity. The *LP* interpretation $\mathbb{M}$ for $\mathscr{L}$ is a pair $< \mathscr{D}, I >$, where $\mathscr{D}$ is a non-empty set and I assigns denotations to the non-logical symbols of $\mathscr{L}$ in the following way.

- For any constant symbol, c, $I(c)$ is a member of $\mathscr{D}$.
- For every n-ary predicate symbol φ, $I(\varphi)$ is the pair $< I^+(\varphi), I^-(\varphi) >$ where $I^+(\varphi)$ and $I^-(\varphi)$ are the extension and anti-extension of φ respectively.

We should note that, for any n-ary predicate φ,
$I^+(\varphi) \cup I^-(\varphi) = \{< c_1, ..., c_n > | c_1, ..., c_n \in \mathscr{D}\}$ but no need to be φ,
$I^+(\varphi) \cup I^-(\varphi) = \emptyset$.

The identity as a congruence relation

Let v be a valuation from the formulas to truth-values where $v(\varphi) \in \{\{1\},\{1,0\},\{0\}\}$. Where φ is a predicate and $t_1,...,t_n$ are terms, the valuations for atomic formulas, negation $(\neg)$, conjunction $(\wedge)$ are as follows:

- For an n-ary predicate φ,

$$1 \in v(\varphi(t_1,...,t_n))) \text{ iff } < I(t_1),...,(t_n) > \in I^+(\varphi)$$
$$0 \in v(\varphi(t_1,...,t_n))) \text{ iff } < I(t_1),...,(t_n) > \in I^-(\varphi)$$

- For a formula Φ, Ψ of $\mathscr{L}$,

$$1 \in v(\neg\Phi) \text{ iff } 0 \in v(\Phi),$$
$$0 \in v(\neg\Phi) \text{ iff } 1 \in v(\Phi),$$
$$1 \in v(\Phi \wedge \Psi) \text{ iff } 1 \in v(\Phi) \text{ and } 1 \in v(\Psi),$$
$$0 \in v(\Phi \wedge \Psi) \text{ iff } 0 \in v(\Phi) \text{ or } 0 \in v(\Psi).$$

The identity as a congruence relation

Truth conditions for classical logic are obtained by ignoring the second clause of each connective. The above interpretation extends to equality in the following way.

Definition

For any given $LP-$interpretation I and J, J is an extension of I iff for every predicate φ, $I^+(\varphi) \subseteq J^+(\varphi)$ and $I^-(\varphi) \subseteq J^-(\varphi)$.

Theorem

Let I, J be (LP-)interpretations and J is an extension of I. Let v_1, v_2 *are valuations for I, J respectively. For any formula Phi,* $v_1(\Phi) \subseteq v_2(\Phi)$.

Proof.

See Priest(1997) □

The identity as a congruence relation

Priest(1997, 2000) takes the names to be the members of $\mathscr{D}$ themselves and adopts the convention that for every $c \in \mathscr{D}, I(c)$ is just c itself. Let $\mathscr{L}_{\mathbb{M}}$ be a language of $\{0, S, +, \cdot\}$ for (inconsistent) arithmetic augmented from $\mathscr{L}$ with a name for every member of $\mathscr{D}$ of $\mathbb{M}$ and Γ be a theory in $\mathscr{L}_{\mathbb{M}}$ extending *PA*. Suppose $\mathscr{M}$ be a non-standard model of Γ and $\equiv$ be a congruence relation on $\mathscr{M}$ with only finitely many equivalence classes.

$$a \equiv b(mod\ d) \text{ if } a = b + dk \text{ for some integer } k.$$

The identity as a congruence relation

We define $\mathscr{D}^{\equiv}$ to be the set of equivalence classes and say $[c]$ is the equivalence class of c in *mathcalD* under $\approx$. So to speak, $[c]$ is defined as $\{x|x \equiv c\}$ and $\mathscr{D}^{\equiv} = \{[c]|c \in \mathscr{D}\}$. A new interpretation, $\mathbb{M}^{\equiv} =< \mathscr{D}^{\equiv}, I^{\equiv} >$, called the *collapsed interpretation* is given as follows:

- For every constant c, $I^{\equiv}(c) = [I(c)]$.
- For every n-place function f, $I^{\equiv}(f)([c_1],...,[c_n]) = [I(f)(c_1,...,c_n)]$.
- For every predicate φ and $1 \leq i \leq n$,

$$
\begin{aligned}
&< [c_1],...,[c_n] > \in I^{\equiv}_{+}(\varphi) \text{ iff for some} \\
&d_i \equiv c_i, < [d_1],...,[d_n] > \in I^{+}(\varphi), \\
&< [c_1],...,[c_n] > \in I^{\equiv}_{-}(\varphi) \text{ iff for some} \\
&d_i \equiv c_i, < [d_1],...,[d_n] > \in I^{-}(\varphi),
\end{aligned}
$$

- $I^{\equiv}_{+}([x] = [y]) = \{< [x],[y] > |x \equiv y\}$,
- $I^{\equiv}_{-}([x] = [y]) = \{< [x],[y] > |x \neq y\}$.

The identity as a congruence relation

- $I_{+}^{\equiv}([x] = [y]) = \{< [x], [y] > | x \equiv y\}$,
- $I_{-}^{\equiv}([x] = [y]) = \{< [x], [y] > | x \neq y\}$.

If x and y are distinct numbers, and in the equivalence class $[z]$, $[x] = [y]$ but $[x] \neq [y]$ since $x \equiv y$ but $x \neq y$. Hence, $\exists x(x = x \wedge x \neq x)$ holds in $\mathbb{M}^{\equiv}$.

The interpretation $I^{\equiv}$ of $\approx$ says that for any natural number x, y, and m, $\overline{x} \approx_m \overline{y}$ is true when $x \equiv y(mod\ m)$ and is false when $x \neq y$. For instance, $\overline{3}$ is identical with $\overline{5}$ with regard to the modulo 2 because $3 \equiv 5(mod\ 2)$ but $\overline{3}$ is not identical with $\overline{5}$ because $3 \neq 5$. Hence, $\exists x \exists y(x \approx y \wedge x \not\approx y)$ for some modulo number m in $\mathbb{M}^{\equiv}$. This shows that his identity relation $\approx$ is not the same as the standard identity $=$. $\approx$ does not represent the identity function.

Question 1. Does Priest's identity relation represent identity function? No!
Question 2. Is Priest's identity relation $\approx$ primitive recursive?

Can Priest's identity relation be both inconsistent and primitive recursive?

- 'x is identical with $y(x \equiv_n y)$' : $rem(x,n) = rem(y,n)$.
- 'x is identical with y' : $x \neq y$.

Base 1. If Priest's identity relation is inconsistent, then it is not primitive recursive in the standard sense.

Base 2. If it is primitive recursive in the standard sense, then it does not have any primitive recursive function f such that it represents f and $f = f \neq f$.

Base 3. Priest did not propose a new definition of 'primitive recursive function' that fits to his dialetheism.

Claim. There is no inconsistent primitive recursive relation. (Because there is no primitive recursive function f in standard sence such that $f = f \neq f$ is true.)

Can Priest's identity relation be both inconsistent and primitive recursive?

- 'x is identical with $y(x \equiv_n y)$' : $rem(x,n) = rem(y,n)$.
- 'x is identical with y' : $x \neq y$.

$rem(x,y) := \begin{cases} the\ remainder\ when\ x\ is\ divided\ by\ y & if \quad y \neq 0. \\ 0 & if \quad y = 0. \end{cases}$ For some n and the divisor d, $rem(x,d) = n$ means that n is the remainder when x is divided by d.

Let us define the characteristic function f:

$f(x,y) := \begin{cases} 1 & if \quad rem(x,d) = rem(y,d) for\ some\ divisor\ d. \\ 0 & \quad otherwise. \end{cases}$

Then, the congruence relation $\equiv$ represents f and $\equiv$ is primitive recursive.

Can Priest's identity relation be both inconsistent and primitive recursive?

$$f(x,y) := \begin{cases} 1 & if \quad rem(x,d) = rem(y,d) for\ some\ divisor\ d. \\ 0 & otherwise. \end{cases}$$

$$g(x,y) := \begin{cases} 1 & if \quad x = y. \\ 0 & otherwise. \end{cases}$$

Suppose that $\approx$ is inconsistent in PA^*. Then, for some natural numbers m and n, $PA^* \vdash \overline{m} \approx \overline{n} \wedge \overline{m} \not\approx \overline{n}$. Then, Priest's identity relation $\approx$ represents f with the value 1 and $\not\approx$ represents g with the value 0. For any natural number m and n, the following relations hold:

$$PA^* \vdash \overline{m} \approx \overline{n} \quad iff \quad f(m,n) = 1. \tag{8}$$

$$PA^* \vdash \overline{m} \not\approx \overline{n} \quad iff \quad g(m,n) = 0. \tag{9}$$

Then, from (9), we have the relation that $PA^* \vdash \overline{m} \approx \overline{n}$ if and only if $g(m,n) = 1$. Hence, by (8), $f(m,n) = 1$ iff $g(m,n) = 1$. However, this does not hold. $\approx$ fails to be a primitive recursive.

Can Priest's identity relation be both inconsistent and primitive recursive?

For (ii), we suppose that $\approx$ is primitive recursive. Then, there is a primitive recursive function h such that $\approx$ represents h in PA^*. That is to say, for any natural number m and n, $PA^* \vdash \overline{m} \approx \overline{n}$ iff $h(m,n) = 1$ and $PA^* \vdash \overline{m} \not\approx \overline{n}$ iff $h(m,n) = 0$. If $\approx$ is inconsistent, then $PA^* \vdash \overline{m} \approx \overline{n} \wedge \overline{m} \not\approx \overline{n}$. It implies that $h(m,n) = 1 = 0$. Therefore, $\approx$ is not inconsistent.

Similarly, we have an argument for the general case that for any relation φ in PA^*, $PA^* \not\vdash \varphi \wedge \neg\varphi$ or φ is not primitive recursive. Suppose that there is φ in the language of PA^* such that $PA^* \vdash \varphi \wedge \neg\varphi$ and φ is primitive recursive. Then, for any natural number $n_1, \ldots, n_k$, there is a primitive recursive function f such that $f(n_1, \ldots, n_k) = 1$ iff $PA^* \vdash \varphi(\overline{n_1}, \ldots, \overline{n_k})$ and $f(n_1, \ldots, n_k) = 0$ iff $PA^* \vdash \neg\varphi(\overline{n_1}, \ldots, \overline{n_k})$. Since $PA^* \vdash \varphi(\overline{n_1}, \ldots, \overline{n_k}) \wedge \neg\varphi(\overline{n_1}, \ldots, \overline{n_k})$, $f(n_1, \ldots, n_k) = 0 = 1$. Hence, there is no φ such that $PA^* \vdash \varphi \wedge \neg\varphi$ and φ is primitive recursive.

Can Priest's identity relation be both inconsistent and primitive recursive?

... if the [primitive] recursive relationships are specified by [PA*], they are not all representable in PA- just because it is consistent. Where does the proof of the fact that all [primitive] recursive relationships are representable in PA break down, however? The answer depends on which proof we are talking about, and on which inconsistent theory of arithmetic is correct. Priest(2006, p. 241)

Happy 2nd Korea Logic Day!!