One-variable theorem for antichain tree property

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Outline

Preliminaries on basic model theory

Definition

- By a *language*, we mean a set of constant symbols, relation symbols, and function symbols.
- Let $\mathcal{L}$ be a language. By an *$\mathcal{L}$-theory*, we mean a set of $\mathcal{L}$-sentences ($\mathcal{L}$-formula without free variable) which yields no contradiction.
- Let $\mathcal{L} = \{c_0, ..., R_0, ..., f_0, ...\}$ be a language, where c_i is a constant symbol for each i, R_i is an n_i-ary relation symbol for each i, and f_i is an m_i-ary function symbol for each i. By an *$\mathcal{L}$-structure (model) $\mathbb{M}$*, we mean a tuple $(M, c_0^{\mathbb{M}}, ..., R_0^{\mathbb{M}}, ..., f_0^{\mathbb{M}}, ...)$, where M is a set, $c_i^{\mathbb{M}} \in M$ for each i, $R_i^{\mathbb{M}} \subseteq M^{n_i}$ for each i, and $f_i : M^{m_k} \to M$ for each i. M is called the universe of $\mathbb{M}$.

Preliminaries on basic model theory

Definition

- Let $\mathcal{L}$ be a language, T be an $\mathcal{L}$-theory, σ be an $\mathcal{L}$-sentence, $\mathbb{M}$ be an $\mathcal{L}$-structure. By $\mathbb{M} \models \sigma$, we mean σ is true in $\mathbb{M}$. By $\mathbb{M} \models T$, we mean that $\mathbb{M} \models \sigma$ for all $\sigma \in T$.
- Let $\mathcal{L}$ be a language, $\mathbb{M}$ be an $\mathcal{L}$-structure. By $Th(\mathbb{M})$, we mean the set of all $\mathcal{L}$-sentences σ such that $\mathbb{M} \models \sigma$.
- Let $\mathcal{L}$ be a language, T be an $\mathcal{L}$-theory. By $Mod(T)$, we mean the class of all $\mathcal{L}$-structures $\mathbb{M}$ such that $\mathbb{M} \models T$.
- Let $\mathcal{L}$ be a language, $\mathcal{K}$ be a class of $\mathcal{L}$-structures. We say $\mathcal{K}$ is an *elementary class* if there exists an $\mathcal{L}$-theory T such that $\mathcal{K} = \mathrm{Mod}(T)$.

Preliminaries on basic model theory

Example

Let $\mathbb{C}$, $\mathbb{R}$, $\mathbb{Q}$, $\mathbb{Z}$ be the set of complex numbers, real numbers, rational numbers, integers, respectively.

- Let $\mathcal{L} = \{0, 1, +, -, \cdot\}$ be a language where 0, 1 are constant symbols, $+$, $-$, $\cdot$ are binary function symbols. Then $\mathbb{C}$ and $\mathbb{R}$ can be regarded as $\mathcal{L}$-structures and $\mathbb{C} \models \sigma$, $\mathbb{R} \models \neg\sigma$ where
$$\sigma := \exists x (x \cdot x = -1).$$
- Let $\mathcal{L} = \{<\}$ be a language where $<$ is a binary relation symbol. Then $\mathbb{Q}$ and $\mathbb{Z}$ can be regarded as $\mathcal{L}$-structures and $\mathbb{Q} \models \tau$, $\mathbb{Z} \models \neg\tau$ where
$$\tau := \forall xy (x < y \to \exists z (x < z < y)).$$

Categorizing first-order theories

We can categorize first-order theories according to the combinatorial configurations. Let $\mathcal{L}$ be a language and T be an $\mathcal{L}$-theory.

Definition

T is said to be *stable* if there is no $\mathcal{L}$-formula $\varphi(x, y)$, $\mathcal{L}$-structure $\mathbb{M} \models T$ and $(a_i, b_i)_{i\in\omega} \in \mathbb{M}$ such that

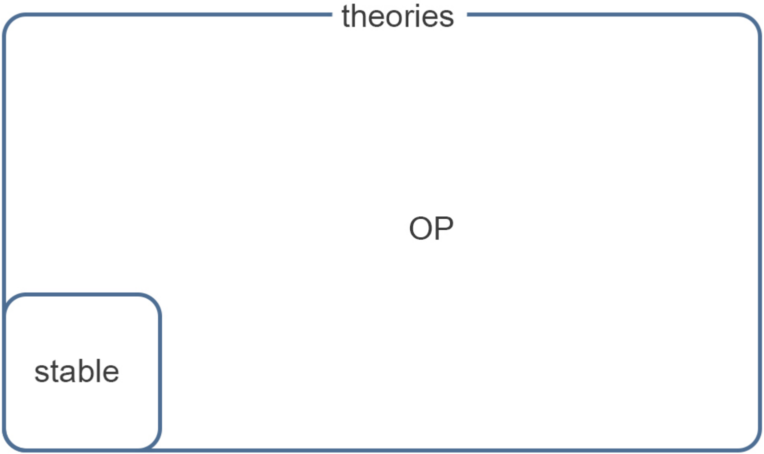
$$(i) \quad \mathbb{M} \models \varphi(a_i, b_j) \text{ if and only if } i < j.$$

If there exist φ, $\mathbb{M} \models T$, $(a_i, b_i) \in \mathbb{M}$ which satisfy (i), then we say T has the *order property (OP)*.

Example

Let $\mathcal{L} = \{<\}$. Then $\mathrm{Th}(\mathbb{Z})$ has the order property (thus it is unstable) since $\varphi(x, y) := x < y$ satisfies (i) with $a_i = 2i, b_i = 2i - 1 \in \mathbb{Z}$.

Categorizing first-order theories



Categorizing first-order theories

Definition

T is said to be *NIP* if there is no $\mathcal{L}$-formula $\varphi(x, y)$, $\mathcal{L}$-structure $\mathbb{M} \models T$, $(a_i)_{i \in \omega} \in \mathbb{M}$, and $(b_I)_{I \subseteq \omega}$ such that

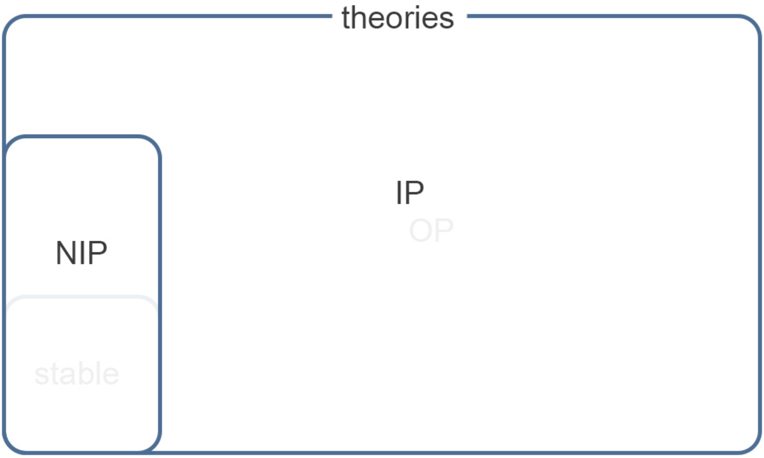
$$(ii) \quad \mathbb{M} \models \varphi(a_i, b_I) \text{ if and only if } i \in I.$$

If there exist φ, $\mathbb{M} \models T$, $(a_i, b_I) \in \mathbb{M}$ which satisfy (ii), then we say T has the *independence property (IP)*.

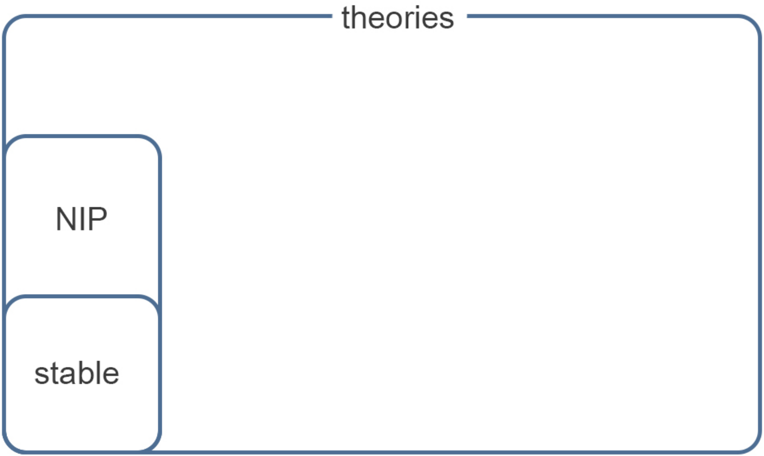
Example

Let $\mathcal{L} = \{0, 1, +, -, \cdot\}$. Then Th($\mathbb{N}$) has the independence property since $\varphi(x, y) := \exists z(x \cdot z = y)$ satisfies (ii) with $a_i = p_i$ and $b_I = \prod_{i \in I} p_i$ where p_i is the i-th prime number.

Categorizing first-order theories



Categorizing first-order theories



stable $\Rightarrow$ NIP

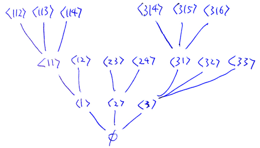
Categorizing first-order theories

Notation

- $\omega^{<\omega}$ is the set of all finite sequences of natural numbers.
- ω^{ω} is the set of all infinite (countable) sequences of natural numbers.
- For $\eta, \nu \in \omega^{<\omega}$, by $\eta \trianglelefteq \nu$ we mean that η is an initial segment of ν.
- For $\eta, \nu \in \omega^{<\omega}$, by $\eta \perp \nu$ we mean that $\eta \ntrianglelefteq \nu$ and $\nu \ntrianglelefteq \eta$.
- We assume $\emptyset \in \omega^{<\omega}$ and $\emptyset \trianglelefteq \eta$ for all $\eta \in \omega^{<\omega}$.

Example

- $\langle 2\rangle \trianglelefteq \langle 2,3\rangle \trianglelefteq \langle 2,3,1\rangle \trianglelefteq \langle 2,3,1,5\rangle \trianglelefteq \ldots$.
- $\langle 7,2,5\rangle \perp \langle 7,2,9\rangle$.



Categorizing first-order theories

Definition

T is said to be *simple* if there is no $\mathcal{L}$-formula $\varphi(x, y)$, $\mathcal{L}$-structure $\mathbb{M} \models T$, $(a_\eta)_{\eta \in \omega^{<\omega}}$ such that

(iii) $\{\varphi(x, a_{\eta \lceil n})\}_{n \in \omega}$ is consistent for all $\eta \in \omega^\omega$.

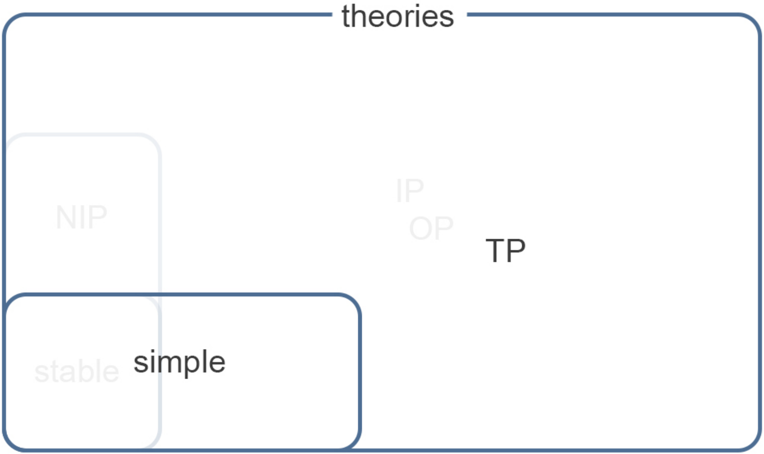
(iv) $\{\varphi(x, a_{\eta \frown i}), \varphi(x, a_{\eta \frown j})\}$ is inconsistent for all $\eta \in \omega^{<\omega}$ and $i < j \in \omega$.

If there exist φ, $\mathbb{M} \models T$, $(a_\eta)_{\eta \in \omega^{<\omega}} \in \mathbb{M}$ which satisfy (iii) and (iv), then we say T has the *tree property (TP)*.

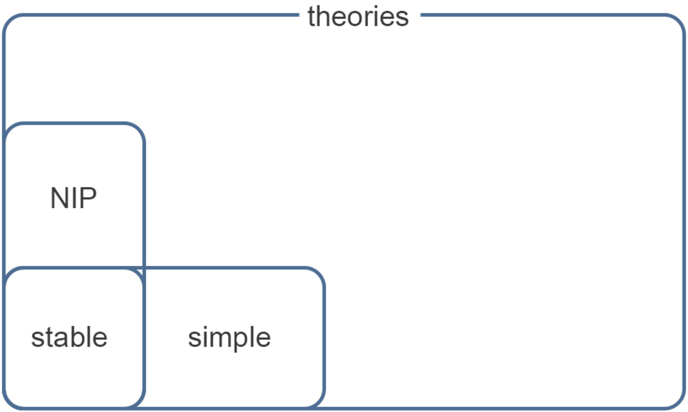
Example

Let $\mathcal{L} = \{<\}$. Then $\mathrm{Th}(\mathbb{R})$ has the tree property since $\varphi(x, y_0, y_1) := y_0 < x < y_1$ satisfies (iii) and (iv) with $a_{\eta \frown i} = (\sum_{k \leq l(\eta)} \frac{1}{10^k}\eta(k) + \frac{1}{10^{l(\eta)+1}} i, \sum_{k \leq l(\eta)} \frac{1}{10^k}\eta(k) + \frac{1}{10^{l(\eta)+1}}(i+1))$.
For example, $a_{\langle 2,3,5 \rangle} = (0.235, 0.236)$ and hence $\varphi(x, a_{\langle 2,3,5 \rangle}) := 0.235 < x < 0.236$.
$\{\varphi(x, a_{\langle 3 \rangle}), \varphi(x, a_{\langle 3,4 \rangle}), \varphi(x, a_{\langle 3,4,8 \rangle}), \ldots\}$ is consistent.
$\{\varphi(x, a_{\langle 6,4,7 \rangle}), \varphi(x, a_{\langle 6,4,8 \rangle})\}$ is inconsistent.

Categorizing first-order theories



Categorizing first-order theories



$$\begin{array}{ccc} \text{NIP} & & \\ \Uparrow & & \\ \text{stable} & \Rightarrow & \text{simple} \end{array}$$

Categorizing first-order theories

Definition

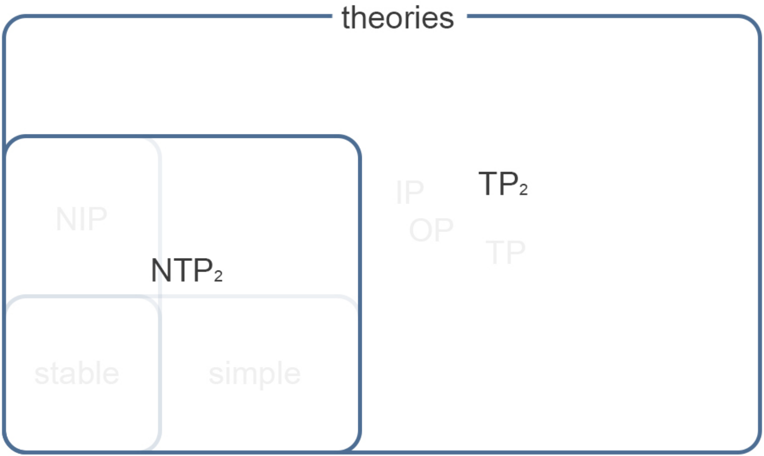
We say T has the *tree property of the second kind (TP_2)* if there exist $\varphi(x,y)$, $\mathbb{M} \models T$, $(a_{i,j})_{i,j\in\omega} \in \mathbb{M}$ such that

$$\{\varphi(x, a_{n,f(n)})\}_{n\in\omega} \text{ is consistent for all } f:\omega\to\omega,$$

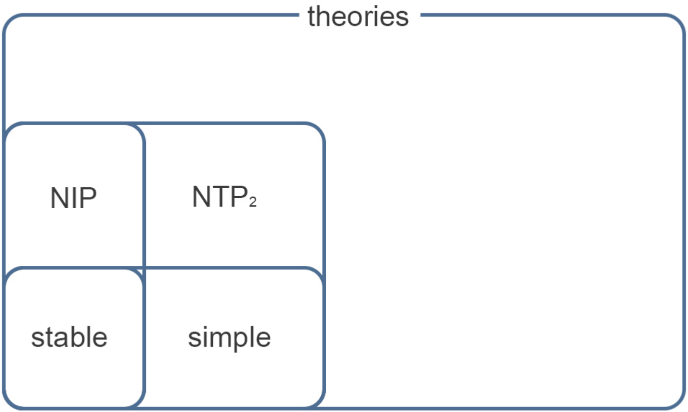
$$\{\varphi(x,a_{i,j}), \varphi(x,a_{i,k})\} \text{ is inconsistent for all } i,j,k\in\omega \text{ with } j\neq k.$$

We say T is *NTP_2* if it does not have TP_2.

Categorizing first-order theories



Categorizing first-order theories



$$\begin{array}{ccc} \text{NIP} & \Rightarrow & \text{NTP}_2 \\ \Uparrow & & \Uparrow \\ \text{stable} & \Rightarrow & \text{simple} \end{array}$$

Categorizing first-order theories

Definition

- We say T has the *tree property of the first kind (TP_1)* if there exist $\varphi(x,y)$, $\mathbb{M} \models T$, $(a_\eta)_{\eta \in \omega^{<\omega}} \in \mathbb{M}$ such that

  $\{\varphi(x, a_{\eta\lceil n})\}_{n\in\omega}$ is consistent for all $\eta \in \omega^\omega$,

  $\{\varphi(x, a_\eta), \varphi(x, a_\nu)\}$ is inconsistent for all $\eta \perp \nu$.

  We say T is *NTP_1* if it does not have TP_1.
- We say T has the *2-strong order property (SOP_2)* if there exist $\varphi(x,y)$, $\mathbb{M} \models T$, $(a_\eta)_{\eta \in 2^{<\omega}} \in \mathbb{M}$ such that

  $\{\varphi(x, a_{\eta\lceil n})\}_{n\in\omega}$ is consistent for all $\eta \in 2^\omega$,

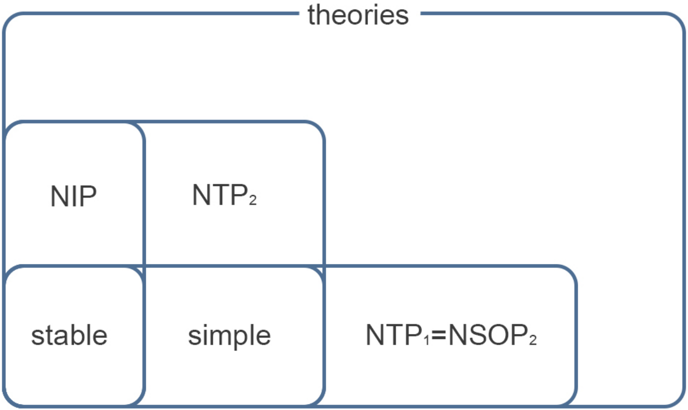
  $\{\varphi(x, a_\eta), \varphi(x, a_\nu)\}$ is inconsistent for all $\eta \perp \nu$.

  We say T is *$NSOP_2$* if it does not have SOP_2.

Fact

- $\mathrm{TP}_1 \Leftrightarrow \mathrm{SOP}_2$.
- $\mathrm{TP} \Leftrightarrow \mathrm{TP}_1 \vee \mathrm{TP}_2$.

Categorizing first-order theories



$$
\begin{array}{ccccc}
\text{NIP} & \Rightarrow & \text{NTP}_2 & & \\
\Uparrow & & \Uparrow & & \\
\text{stable} & \Rightarrow & \text{simple} & \Rightarrow \text{NTP}_1 & \Leftrightarrow \text{NSOP}_2
\end{array}
$$

Categorizing first-order theories

Definition

- We say T has the *1-strong order property (SOP_1)* if there exist $\varphi(x, y)$, $\mathbb{M} \models T$, $(a_\eta)_{\eta \in 2^{<\omega}} \in \mathbb{M}$ such that

  $\{\varphi(x, a_{\eta \restriction n})\}_{n \in \omega}$ is consistent for all $\eta \in 2^\omega$,

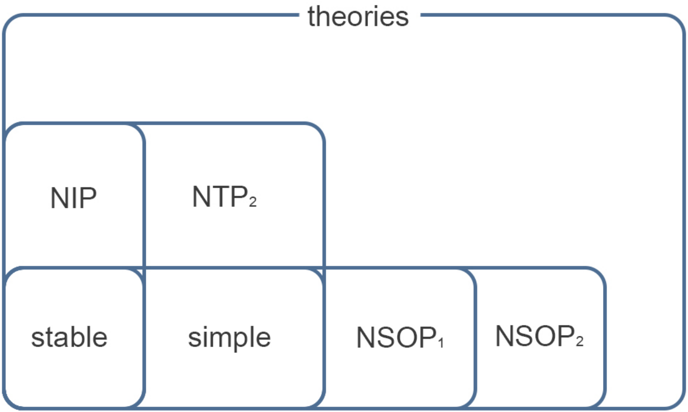
  $\{\varphi(x, a_{\eta \frown 1}), \varphi(x, a_{\eta \frown 0 \frown \nu})\}$ is inconsistent for all $\eta, \nu \in 2^{<\omega}$.

  We say T is *$NSOP_1$* if it does not have SOP_1.

Remark

- simple $\Rightarrow$ NSOP_1.
- $\mathrm{SOP}_2 \Rightarrow \mathrm{SOP}_1$ is well-known. We still do not know whether the converse holds.

Categorizing first-order theories



$$\begin{array}{ccccccc}
\text{NIP} & \Rightarrow & \text{NTP}_2 & & & & \\
\Uparrow & & \Uparrow & & & & \\
\text{stable} & \Rightarrow & \text{simple} & \Rightarrow & \text{NSOP}_1 & \Rightarrow & \text{NSOP}_2
\end{array}$$

Categorizing first-order theories

Definition

- $X \subseteq 2^{<\omega}$ is called an *antichain* if $\eta \perp \nu$ for all distinct $\eta, \nu \in X$.
- We say T has the *antichain tree property (ATP)* if there exist $\varphi(x, y)$, $\mathbb{M} \models T$, $(a_\eta)_{\eta \in 2^{<\omega}} \in \mathbb{M}$ such that

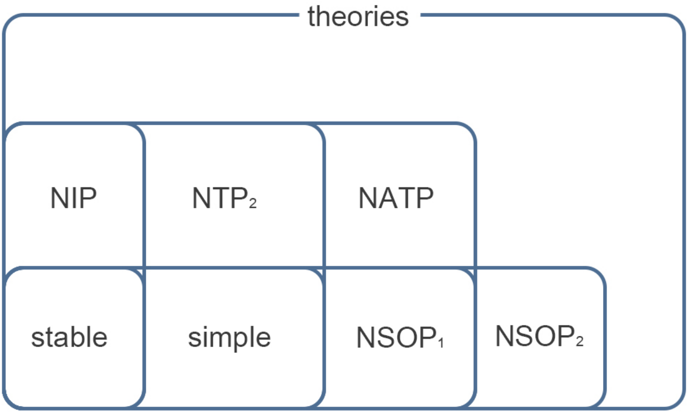
  $\{\varphi(x, a_\eta)\}_{\eta \in X}$ is consistent for all antichain X in $2^{<\omega}$,
  $\{\varphi(x, a_\eta), \varphi(x, a_\nu)\}$ is inconsistent for all $\eta \trianglelefteq \nu$.

  We say T is *NATP* if it does not have ATP.

Remark

- $\mathrm{NSOP}_1 \Rightarrow$ NATP.
- $\mathrm{NTP}_2 \Rightarrow$ NATP.
- If there exists a theory which is SOP_1 and NSOP_2, then the theory is ATP.

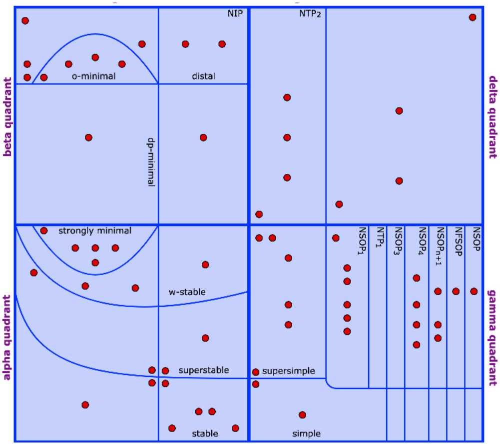
Categorizing first-order theories



$$\begin{array}{ccccccc}
\text{NIP} & \Rightarrow & \text{NTP}_2 & \Rightarrow & \text{NATP} & & \\
\Uparrow & & \Uparrow & & \Uparrow & & \\
\text{stable} & \Rightarrow & \text{simple} & \Rightarrow & \text{NSOP}_1 & \Rightarrow & \text{NSOP}_2
\end{array}$$

Categorizing first-order theories

In fact, there are more dividing lines in the class of first-order theories.



Categorizing first-order theories

This classification gives us ways of understanding mathematical objects, for example

- If a field is superstable, then it is algebraically closed field.
- If a field is of finite dp-rank, then it is perfect.
- If a graph has the tree property, then it is not a random graph.

and so on.

One-variable theorem for antichain tree property

If a theory has TP_1 (or TP_2, SOP_1), then it is witnessed by a formula $\varphi(x, y)$ and the arity of x may vary. Thus if we want to show that a theory is NTP_1 (NTP_2, NSOP_1, respectively) directly from the definition, then we need to check that there is no formula which witnesses TP_1 (TP_2, SOP_1). But the complexity of formula increases rapidly as the arity of its free variable increases. This is the difficulty of showing a theory is NTP_1 (NTP_2, NSOP_1) directly from the definition. One-variable theorem for tree properties may simplify this problem.

One-variable theorem for TP_2 [A. Chernikov]

If T has TP_2, then it is witnessed by $\varphi(x, y)$ with $|x| = 1$.

One-variable theorem for SOP_1 [N. Ramsey]

If T has SOP_1, then it is witnessed by $\varphi(x, y)$ with $|x| = 1$.

One-variable theorem for TP_1 [A. Chernikov, N. Ramsey]

If T has TP_1, then it is witnessed by $\varphi(x, y)$ with $|x| = 1$.

One-variable theorem for antichain tree property

Thus if we want to check a theory is NTP_1 (NTP_2, NSOP_1), then we only need to check that every formula in a single free variable does not witnesses TP_1 (TP_2, SOP_1). Thus it is natural to ask whether the similar statement holds for ATP.

Theorem [J. Ahn, J. Lee, J. Kim]

If T has ATP, then it is witnessed by $\varphi(x, y)$ with $|x| = 1$.

Thus when we check if a theory is NATP, it is enough to show that there is no formula in a single free variable which witnesses ATP. Furthermore, if the theory has the quantifier elimination, then the verification will be much easier by the following observation.

Proposition

If $\varphi \vee \psi$ does not witness ATP, then φ and ψ do not witness ATP.

Therefore we only need to check if there is no formula in a single free variable, of the form the conjunction of basic formulas.

Strategy of the proof

Path-Collapse Lemma [A. Chernikov, N. Ramsey]

Suppose κ is an infinite cardinal, $(a_\eta)_{\eta \in 2^{<\kappa}}$ is a tree str-indiscernible over a set of parameters C and, moreover, $(a_{0^\alpha})_{0<\alpha<\omega}$ is order indiscernible over cC. Let

$$p(y;\bar{z}) = \mathrm{tp}(c;(a_{0\frown 0^\gamma})_{\gamma<\kappa}/C).$$

Then if

$$p(y:(a_{0\frown 0^\gamma})_{\gamma<\kappa}) \cup p(y:(a_{1\frown 0^\gamma})_{\gamma<\kappa})$$

is not consistent, then T has SOP_2, witnessed by a formula with free variables y.

To obtain a witness of ATP in a single free variable, we find an appropriate statement which is similar to the path-collapse lemma. In short, we prove two modified lemmas of the path-collapse lemma for the purpose of dealing with antichains and ATP. The shapes of the lemmas will be made to reflect the construction of antichain trees.

Strategy of the proof

Definition

An antichain $X \subseteq 2^{<\kappa}$ is said to be *maximal* if $Y \subseteq 2^{<\kappa}$ is not an antichain whenever $X \subsetneq Y$.

Remark

Let α_n denotes the number of all maximal antichains in $2^{<n}$. Then $\alpha_0 = 0$ and $\alpha_{n+1} = \alpha_n^2 + 1$ for each $n \in \omega$. Let $\{X_i\}_{i \in \alpha_n}$ be the set of all maximal antichains in $2^{<n}$. Then

$$\{(\langle 0 \rangle ^\frown X_i) \cup (\langle 1 \rangle ^\frown X_j) : i, j < \alpha_n\} \cup \{\emptyset\}$$

is the set of all maximal antichains in $2^{<n+1}$. Thus $\alpha_{n+1} = \alpha_n^2 + 1$ for each $n \in \omega$.

Note that to obtain all maximal antichains in $2^{<n+1}$, first we take the product of two copies of all maximal antichains in $2^{<n}$, and then we add one more maximal antichain which is located below all antichains constructed in the first step.

Strategy of the proof

By a collapsible family of antichains, we mean a set of antichains such that the union of types over each antichain is consistent, or it yields ATP. More precisely,

Definition

Let κ be an infinite cardinal and $X_0, \ldots, X_n \subseteq \mathbb{Q}^{<\kappa}$ be endless dense universal antichains with $|X_0| = \ldots = |X_n|$ and $X_0 \sim_{str} \ldots \sim_{str} X_n$. Let us consider the following condition.

(∗) For any set $C \subseteq \mathbb{M}$, $b \in \mathbb{M}$, tree indexed set $(a_\eta)_{\eta \in \mathbb{Q}^{<\kappa}}$ which is str-indiscernible over C, and $i \leq n$, if $(a_\eta)_{\eta \in X_i}$ is δ-indiscernible over bC, then $\bigcup_{j \leq n} p(y, (a_\eta)_{\eta \in X_j})$ is consistent where $p(y, \bar{z}) = \mathrm{tp}(b, (a_\eta)_{\eta \in X_i} / C)$ or there is a formula with free variable y which witnesses ATP.

If $X_0, \ldots, X_n$ satisfies $(*)$, then we say they are *collapsible*. By a *collapsible family*, we mean a set of endless dense universal antichains which are collapsible.

Strategy of the proof

1st Antichain-Collapse Lemma

Let κ be a sufficiently large cardinal, and $\{X_0, \ldots, X_n\}$ be a collapsible family in $\mathbb{Q}^{<\kappa}$. Then for any $\nu, \xi \in \mathbb{Q}^{<\kappa}$ with $\nu \perp \xi$ and $\nu <_{lex} \xi$,

$$\{X_i^\nu \cup Y \cup X_j^\xi : i, j \leq n\}$$

is a collapsible family, where $X_i^\nu = \nu^\frown X_i$ and $X_j^\xi = \xi^\frown X_j$ for each $i, j \leq n$, and

$$Y = \{\nu \wedge \xi^\frown \langle s \rangle^\frown \eta : \nu(l(\nu \wedge \xi) + 1) < s < \xi(l(\nu \wedge \xi) + 1), \eta \in \mathbb{Q}^\omega\}.$$

Roughly speaking, the first antichain-collapse lemma says that the class of collapsible family is *closed under the product*. In other words, if a set of antichains is collapsible, then the product of two copies of the set is also collapsible.

Strategy of the proof

2nd Antichain-Collapse Lemma

Let κ be a sufficiently large cardinal, and $\{X'_0, ..., X'_n\}$ be a collapsible family in $\mathbb{Q}^{<\kappa}$. Then there is a collapsible family $\{X_0, ..., X_{n+1}\}$ in $\mathbb{Q}^{<\kappa}$ which satisfies that there are $\chi \in X_{n+1}$ and $\chi' \trianglerighteq \chi$ such that $X_0 = \chi'^\frown X'_0, ..., X_n = \chi'^\frown X'_n$.

The second antichain-collapse lemma says that if a collapsible family $\mathcal{F}$ is given, then we can find an appropriate antichain X which is located below $\mathcal{F}$, and $\mathcal{F} \cup \{X\}$ is still collapsible.

Strategy of the proof

Sketch of the proof of the main theorem

Let $\varphi(x, y; z)$ witnesses ATP in free variables x, y where $|y| = 1$. Then by using antichain-collapse lemmas, we can construct collapsible family $\mathcal{F}_n$ for each $n \in \omega$, whose antichains represent all maximal antichains in some binary tree with height n. By choosing suitable elements in antichains of $\mathcal{F}_n$ we can find a common realization for y which makes $\varphi(x; y, z)$ witnesses ATP in free variable x. Thus we can reduce the arity of free variable of witness of ATP and by repeating this we obtain a witness of ATP in a single free variable.

Question

- Is there an example of strictly NATP theory? (an NATP theory which is not NTP_1, not NSOP_1)
- Is there a suitable independence notion for NATP?
- Is there a Kim-Pillay style criterion for NATP?

References

[1] Kota Takeuchi and Akito Tsuboi, On the existence of indiscernible trees, Annals of Pure and Applied Logic, 2012.

[2] Artem Chernikov and Nicholas Ramsey, On model-theoretic tree properties, Journal of Mathematical Logic, (2016).

[3] Byunghan Kim and Hyeung-Joon Kim, Notions around tree property 1, Annals of Pure and Applied Logic 162 (2011).

[4] Lynn Scow, Indiscernibles, EM-types, and Ramsey classes of trees, Notre Dame Journal of Formal Logic, (2015).

[5] Artem Chernikov, Theories without tree tree property of the second kind, Annals of Pure and Applied Logic, (2013)

[6] Nicholas Ramsey, A note on NSOP_1 in one variable, The Journal of Symbolic Logic, (2019).

[7] Wilfrid Hodges, Model theory, Cambridge University Press, Cambridge, (1993).