The Counterexample to the Proof-Theoretic Conjecture for Self-Referential Paradoxes

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History

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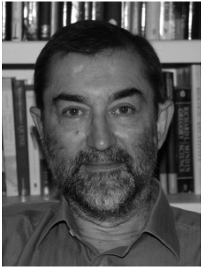
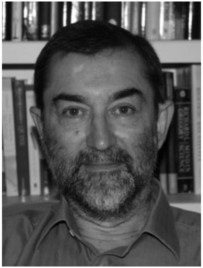


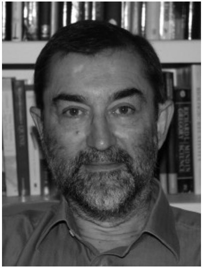
Neil Tennant(1982, 1995, 2015, 2016, 2017) has regarded the non-terminating reduction sequence as the primary feature of paradoxes and proposed the proof-theoretic criterion for paradoxicality.

History



Tennant(1982) sets his criterion for paradoxicality such that the derivations formalizing paradoxes in natural deduction are distinguished by having non-terminating reduction sequences of the derivation of an absurdity ($\perp$) involved. He called a non-terminating reduction sequence a 'looping reduction sequence.'

History



While investigating a non-self-referential paradox suggested by Stephen Yablo(1993), Tennant(1995) has extended his criterion by embracing a spiraling reduction generated by Yablo's paradox. He thought that a looping reduction sequence is the main feature of the self-referential paradoxes but not that of Yablo's paradox. He claimed that the non-terminating reduction sequence enters into loops if the self-reference is involved; otherwise it does not. We interpret his claim as an informal conjecture for self-referential paradoxes that every derivation formalizing a self-referential paradox in natural deduction generates a looping reduction sequence but not a spiraling reduction sequence.

The Proof-Theoretic Criterion for Paradoxicality(*PCP*): Let $\mathfrak{D}$ be any derivation of a given natural deduction system S. $\mathfrak{D}$ is a *T-paradox* if and only if

(i) $\mathfrak{D}$ is a (closed or open) derivation of $\perp$,

(ii) *id est* inferences (or rules) are used in $\mathfrak{D}$,

(iii) a reduction procedure of $\mathfrak{D}$ generates a non-terminating reduction sequence, such as a reduction loop.

The Proof-Theoretic Conjecture for Self-Referential Paradoxes(*PCSP*): Let $\mathfrak{D}$ be any derivation satisfying *PCP*, i.e. a T-paradox. $\mathfrak{D}$ generates a looping reduction sequence if and only if $\mathfrak{D}$ formalizes a self-referential paradox.

Counterexamples and Suggestions

The Aim: To find a correct proof-theoretic structure of self-referential paradoxes.

The Counterexample to *PCSP*: There is a derivation of a self-referential paradox, e.g. the Liar paradox, which satisfies *PCP* but generate a spiraling reduction sequence.

Suggestion: It should be discussed which reduction procedure is admissible in order to find a correct proof-theoretic structure for (self-referential) paradoxes.

Preliminaries

- **Language:**

  Functions and constants: the constant 0, the unary function symbol s for successors, the binary function symbol $+$ for addition.

  Logical operators : $\rightarrow$, $\perp$, and $\forall$ for implication, absurdity, and universal quantification respectively.

  Formulas(predicates) : φ, ψ, and σ for arbitrary formula. Also, the binary predicate $<$ for less-than-relation, and $=$ for equality.

  Negation ($\neg$) : $\neg\varphi$ is defined by $\varphi \rightarrow \perp$.

- **Derivation symbol:** $\mathfrak{D}$

Preliminaries

- **Rules:**

$$\dfrac{\begin{matrix}[\varphi]^1\\ \mathfrak{D}_1\\ \psi\end{matrix}}{\varphi \to \psi} \to I_{,1} \qquad \dfrac{\varphi \to \psi \quad \begin{matrix}\mathfrak{D}_2\\ \varphi\end{matrix}}{\psi} \to E \qquad \dfrac{\begin{matrix}\mathfrak{D}_1\\ \varphi[y/x]\end{matrix}}{\forall x \varphi(x)} \forall I \qquad \dfrac{\forall x \varphi(x)}{\varphi[t/x]} \forall E \qquad \dfrac{\begin{matrix}[\neg\varphi]^1\\ \mathfrak{D}\\ \bot\end{matrix}}{\varphi} CR_{,1}$$

- **Reductions:**

$$\dfrac{\dfrac{\begin{matrix}[\varphi]^1\\ \mathfrak{D}_1\\ \psi\end{matrix}}{\varphi \to \psi} \to I_{,1} \quad \begin{matrix}\mathfrak{D}_2\\ \varphi\end{matrix}}{\psi} \to E \quad \rhd_{\to} \quad \begin{matrix}\mathfrak{D}_2\\ \varphi\\ \mathfrak{D}_1\\ \psi\end{matrix} \qquad \dfrac{\dfrac{\begin{matrix}\mathfrak{D}_1\\ \varphi(y)\end{matrix}}{\forall x \varphi[x/y]} \forall I}{\varphi[t/x]} \forall E \quad \rhd_{\forall} \quad \begin{matrix}\mathfrak{D}_1\\ \varphi[t/y]\end{matrix}$$

Preliminaries

Definitions

- A derivation is in normal if it has no maximum formula.
- A maximum formula is a conclusion of I-rule and a major premise of E-rule.

$$\dfrac{\dfrac{\begin{matrix}[\varphi]^1\\ \mathfrak{D}_1\\ \psi\end{matrix}}{\varphi\to\psi}\to I_{,1} \qquad \begin{matrix}\mathfrak{D}_2\\ \varphi\end{matrix}}{\psi}\to E \qquad \mathit{reduces\ \ to} \qquad \begin{matrix}\mathfrak{D}_2\\ \varphi\\ \mathfrak{D}_1\\ \psi\end{matrix}$$

Preliminaries

Definition

A sequence $< \mathfrak{D}_1, \ldots, \mathfrak{D}_i, \mathfrak{D}_{i+1}, \ldots >$ of derivations is a *reduction sequence* relative to $\mathbb{R}$ iff $\mathfrak{D}_i \rhd \mathfrak{D}_{i+1}$ relative to $\mathbb{R}$ where $1 \leqslant i$ for any natural number i. A derivation $\mathfrak{D}_1$ is *reducible* to $\mathfrak{D}_i$ ($\mathfrak{D}_1 \succ \mathfrak{D}_i$) relative to $\mathbb{R}$ iff there is a sequence $< \mathfrak{D}_1, \mathfrak{D}_2, \ldots, \mathfrak{D}_i >$ relative to $\mathbb{R}$ where for each $j < i$, $\mathfrak{D}_j \rhd \mathfrak{D}_{j+1}$; $\mathfrak{D}_1$ is *irreducible* relative to $\mathbb{R}$ iff there is no derivation $\mathfrak{D}'$ to which $\mathfrak{D}_1 \rhd \mathfrak{D}'$ relative to $\mathbb{R}$ except $\mathfrak{D}_1$ itself.

Definition

A derivation $\mathfrak{D}$ is *normal* (or in *normal form*) relative to $\mathbb{R}$ iff $\mathfrak{D}$ is irreducible relative to $\mathbb{R}$, i.e. $\mathfrak{D}$ has no maximum formula. A reduction sequence *terminates* iff it has a finite number of derivations and its last derivation is in normal form. A derivation $\mathfrak{D}$ is *normalizable* relative to $\mathbb{R}$ iff there is a terminating reduction sequence relative to $\mathbb{R}$ starting from $\mathfrak{D}$.

The Proof-Theoretic Criterion for Paradoxicality(PCP)

- **Formalizing the Liar Paradox in Natural Deduction:** We use an unary truth-predicate $T(\ulcorner x \urcorner)$ which states that x is true. Let us define Φ be a sentence $\neg T(\ulcorner \Phi \urcorner)$. We call Φ a liar sentence. The so-called liar sentence says: "This sentence is not true."

$$
\begin{array}{ccc}
 & \Phi & \\
\nearrow & & \searrow \\
T(\ulcorner \Phi \urcorner) & & \neg T(\ulcorner \Phi \urcorner) \\
\nwarrow & & \swarrow \\
 & \Phi &
\end{array}
$$

The Proof-Theoretic Criterion for Paradoxicality(PCP)

- **Formalizing the Liar Paradox in Natural Deduction:** We use rules for $T(\ulcorner x \urcorner)$ which states that x is true.

$$\frac{\varphi}{T(\ulcorner \varphi \urcorner)}\, TI \qquad \frac{T(\ulcorner \varphi \urcorner)}{\varphi}\, TE$$

- The standaed reduction procedure for $T(x)$ is as below.

$$\dfrac{\dfrac{\begin{matrix}\mathfrak{D}\\ \varphi\end{matrix}}{T(\ulcorner \varphi \urcorner)}\, TI}{\varphi}\, TE \qquad \rhd_{T(x)} \qquad \begin{matrix}\mathfrak{D}\\ \varphi\end{matrix}$$

The Proof-Theoretic Criterion for Paradoxicality(PCP)

- **Formalizing the Liar Paradox in Natural Deduction:** For formulating the Liar paradox, S_L has a Tennant's rules for the liar sentence Φ from Tennant(2016, 2017).

$$\dfrac{\begin{array}{c}[T(\ulcorner\Phi\urcorner)]^1\\ \mathfrak{D}_1\\ \bot\end{array}}{\Phi}\,\Phi I_{,1} \qquad \dfrac{\Phi \qquad \begin{array}{c}[\neg T(\ulcorner\Phi\urcorner)]^1\\ \mathfrak{D}_2\\ \varphi\end{array}}{\varphi}\,\Phi E_{,1}$$

- The reduction procedure for Φ as follows.

$$\dfrac{\dfrac{\begin{array}{c}[T(\ulcorner\Phi\urcorner)]^1\\ \mathfrak{D}_1\\ \bot\end{array}}{\Phi}\,\Phi I_{,1} \qquad \begin{array}{c}[\neg T(\ulcorner\Phi\urcorner)]^2\\ \mathfrak{D}_2\\ \varphi\end{array}}{\varphi}\,\Phi E_{,2} \quad \rhd_{\Phi} \quad \begin{array}{c}\dfrac{\begin{array}{c}[T(\ulcorner\Phi\urcorner)]^1\\ \mathfrak{D}_1\\ \bot\end{array}}{\neg T(\ulcorner\Phi\urcorner)}\,\to I_{,1}\\ \mathfrak{D}_2\\ \varphi\end{array}$$

The Proof-Theoretic Criterion for Paradoxicality(PCP)

- **Formalizing the Liar Paradox in Natural Deduction:** First, we have an open derivation Σ_1 of $\bot$ from $[T(\ulcorner\Phi\urcorner)]$ below left. With Σ_1, there is a closed derivation Σ_2 of $T(\ulcorner\Phi\urcorner)$ below right.

$$\dfrac{\dfrac{[T(\ulcorner\Phi\urcorner)]^1}{\Phi}\,TE \qquad \dfrac{[\neg T(\ulcorner\Phi\urcorner)]^2 \quad [T(\ulcorner\Phi\urcorner)]^1}{\bot}\to E}{\bot}\,\Phi E_{,2} \qquad\qquad \dfrac{\dfrac{\begin{matrix}[T(\ulcorner\Phi\urcorner)]^1\\ \Sigma_1\\ \bot\end{matrix}}{\Phi}\,\Phi I_{,1}}{T(\ulcorner\Phi\urcorner)}\,TI$$

- Then, we have a closed derivation Σ_3 of $\bot$.

$$\dfrac{\dfrac{\begin{matrix}[T(\ulcorner\Phi\urcorner)]^1\\ \Sigma_1\\ \bot\end{matrix}}{\neg T(\ulcorner\Phi\urcorner)}\to I_{,1} \qquad \begin{matrix}\Sigma_2\\ T(\ulcorner\Phi\urcorner)\end{matrix}}{\bot}\to E$$

- **Formalizing the Liar Paradox in Natural Deduction:** $\neg T(\ulcorner \Phi \urcorner)$ in the last $\rightarrow E-$rule of Σ_3 is a maximum formula. Σ_3 reduces to the derivation Σ_4 above. By applying $\rhd_{T(x)}$ and $\rhd_{\Phi}$ to Σ_4, we have the same derivation with Σ_3. Σ_3 generates a non-terminating reduction sequence and thus it is not normalizable.

$$\dfrac{\dfrac{\dfrac{\dfrac{\begin{matrix}[T(\ulcorner \Phi \urcorner)]^1 \\ \Sigma_1 \\ \bot\end{matrix}}{\Phi}\ \Phi I_{,1}}{T(\ulcorner \Phi \urcorner)}\ TI}{\Phi}\ TE \qquad \dfrac{[\neg T(\ulcorner \Phi \urcorner)]^2 \qquad \dfrac{\dfrac{\begin{matrix}[T(\ulcorner \Phi \urcorner)]^3 \\ \Sigma_1 \\ \bot\end{matrix}}{\Phi}\ \Phi I_{,3}}{T(\ulcorner \Phi \urcorner)}\ TI}{\bot}\rightarrow E}{\bot}\ \Phi E_{,2}$$

$$\dfrac{\dfrac{\begin{matrix}[T(\ulcorner \Phi \urcorner)]^1 \\ \Sigma_1 \\ \bot\end{matrix}}{\neg T(\ulcorner \Phi \urcorner)}\rightarrow I_{,1} \qquad \begin{matrix}\Sigma_2 \\ T(\ulcorner \Phi \urcorner)\end{matrix}}{\bot}\rightarrow E$$

The Proof-Theoretic Criterion for Paradoxicality(PCP)

The Proof-Theoretic Criterion for Paradoxicality(PCP): Let $\mathfrak{D}$ be any derivation of a given natural deduction system S. $\mathfrak{D}$ is a *T-paradox* if and only if

(i) $\mathfrak{D}$ is a (closed or open) derivation of $\perp$,

(ii) *id est* inferences (or rules) are used in $\mathfrak{D}$,

(iii) a reduction procedure of $\mathfrak{D}$ generates a non-terminating reduction sequence, such as a reduction loop.

- **Formalizing Yablo's paradox in Natural Deduction:** Yablo's paradox begins with an infinite sequence of sentences $S_1, S_2, S_3, \ldots$, each to the effect that every subsequent sentence is not true.

  (S_1) for all $u > 1$, S_u is not true,
  (S_2) for all $u > 2$, S_u is not true,
  (S_3) for all $u > 3$, S_u is not true, ...

The Conjecture for Self-Referential Paradoxes

- **Formalizing Yablo's paradox in Natural Deduction:** we define S_v as $\forall x(x > v \rightarrow \neg T(\ulcorner S_x \urcorner))$.

$$
\begin{array}{ccc}
 & T(\ulcorner S_u \urcorner) & \\
\swarrow & & \searrow \\
\forall x(x > u \rightarrow \neg T(\ulcorner S_x \urcorner)) & & \forall x(x > u \rightarrow \neg T(\ulcorner S_x \urcorner)) \\
\downarrow & & \downarrow \\
u+1 > u \rightarrow \neg T(\ulcorner S_{u+1} \urcorner)) & & \forall x(x > u+1 \rightarrow \neg T(\ulcorner S_{u+1} \urcorner)) \\
\downarrow & & \downarrow \\
\neg T(\ulcorner S_{u+1} \urcorner) & & T(\ulcorner S_{u+1} \urcorner)
\end{array}
$$

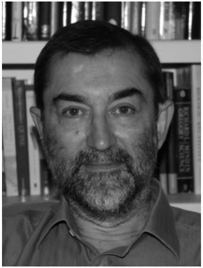
The Conjecture for Self-Referential Paradoxes

- **Proposition 2.4.** There is a closed derivation of $\bot$ in S_Y with respect to $\mathbb{R}_Y$, which initiates a non-terminating reduction sequence and so is not normalizable.

$$
\begin{array}{cc}
\begin{array}{c}
[T(\ulcorner S_w \urcorner)]^1 \\
\theta(w) \\
\bot \\
\hline
\neg T(\ulcorner S_w \urcorner) \quad {\scriptstyle \rightarrow I_{,1}} \\
\hline
w > v \rightarrow \neg T(\ulcorner S_w \urcorner) \quad {\scriptstyle \rightarrow I_{,\emptyset}} \\
\hline
\forall x(x > v \rightarrow \neg T(\ulcorner S_x \urcorner)) \quad {\scriptstyle \forall I} \\
\cdots\cdots\cdots\cdots\cdots \quad {\scriptstyle def} \\
S_v \\
\hline
T(\ulcorner S_v \urcorner) \quad {\scriptstyle TI} \\
\theta(v) \\
\bot
\end{array}
&
\begin{array}{c}
[T(\ulcorner S_w \urcorner)]^1 \\
\theta(w) \\
\bot \\
\hline
\neg T(\ulcorner S_w \urcorner) \quad {\scriptstyle \rightarrow I_{,1}} \\
\hline
w > v+1 \rightarrow \neg T(\ulcorner S_w \urcorner) \quad {\scriptstyle \rightarrow I_{,\emptyset}} \\
\hline
\forall x(x > v+1 \rightarrow \neg T(\ulcorner S_x \urcorner)) \quad {\scriptstyle \forall I} \\
\cdots\cdots\cdots\cdots\cdots \quad {\scriptstyle def} \\
S_{v+1} \\
\hline
T(\ulcorner S_{v+1} \urcorner) \quad {\scriptstyle TI} \\
\theta(v+1) \\
\bot
\end{array}
\end{array}
$$

- Tennant called the abovereduction sequence a spiraling reduction sequence.

The Conjecture for Self-Referential Paradoxes



I shall make so bold as to suggest that it is precisely when the non-terminating reduction procedures enter loops that self-reference is involved. And when they don't enter loops - as with Yablo's example - then self-reference is not involved. (Tennant(1995, p. 207).)

The Conjecture for Self-Referential Paradoxes

The Proof-Theoretic Conjecture for Self-Referential Paradoxes(*PCSP*): Let $\mathfrak{D}$ be any derivation satisfying *PCP*, i.e. a T-paradox. $\mathfrak{D}$ generates a looping reduction sequence if and only if $\mathfrak{D}$ formalizes a self-referential paradox.

The Counterexample : There is a derivation which generates a spiraling reduction sequence but formalizes the Liar paradox.

The Counterexample to *PCSP*

- An additional reduction procedure for *classical reductio* introduced by Gunnar Stålmark (1991, pp. 131-132).

$$\dfrac{\dfrac{\begin{array}{c}[\neg\varphi]^1\\ \mathfrak{D}_1\\ \bot\end{array}}{\varphi}CR_{,1} \qquad \begin{array}{c}\mathfrak{D}_2\\ \psi\end{array}}{\sigma}\circ E \qquad \rhd_{CR(\circ)} \qquad \dfrac{\begin{array}{c}\dfrac{[\neg\sigma]^2 \qquad \dfrac{[\varphi]^1 \qquad \begin{array}{c}\mathfrak{D}_2\\ \psi\end{array}}{\sigma}\circ E}{\bot}\to I_{,1}\\ \mathfrak{D}_1\\ \bot\end{array}}{\sigma}CR_{,2}$$

Proposition 3.1. *There is a closed derivation of $\bot$ in S_{CL} with respect to $\mathbb{R}_{CL}$, which generates a non-terminating reduction sequence and so is irreducible.*

Proof. Two claims will verify the result.

Claim 1. There is a closed derivation Π_3 of $\bot$.

There is an open derivation Π_1 of $\bot$ from $[\neg\Phi]$ as left below. With Π_1, there is an open derivation Π_2 of $\bot$ from $[\neg T(\ulcorner\Phi\urcorner)]$ as right below.

$$\dfrac{[\neg\Phi]^2 \qquad \dfrac{\dfrac{[\neg\Phi]^2 \qquad \dfrac{[T(\ulcorner\Phi\urcorner)]^1}{\Phi}\,TE}{\bot}\to E}{\Phi}\,\Phi I_{,1}}{\bot}\to E \qquad\qquad \dfrac{[\neg T(\ulcorner\Phi\urcorner)]^3 \qquad \dfrac{\dfrac{\begin{matrix}[\neg\Phi]^2\\ \Pi_1\\ \bot\end{matrix}}{\Phi}\,CR_{,2}}{T(\ulcorner\Phi\urcorner)}\,TI}{\bot}\to E$$

Having Π_1 and Π_2, there is a closed derivation Π_3 of $\bot$.

$$\dfrac{\dfrac{\begin{matrix}[\neg\Phi]^4\\ \Pi_1\\ \bot\end{matrix}}{\Phi}\,CR_{,4} \qquad \begin{matrix}[\neg T(\ulcorner\Phi\urcorner)]^3\\ \Pi_2\\ \bot\end{matrix}}{\bot}\,\Phi E_{,3}$$

Claim 2. Π_3 generates a non-terminating reduction sequence and so is irreducible with respect to $\mathbb{R}_{CL}$.

Π_3 is reducible to the following derivation Π_4 by $\rhd_{CR(\Phi)}$.

$$\dfrac{\begin{matrix}\dfrac{[\neg\bot]^5 \qquad \dfrac{[\Phi]^6 \qquad \begin{matrix}[\neg T(\ulcorner\Phi\urcorner)]^3\\ \Pi_2\\ \bot\end{matrix}}{\bot}\,\Phi E_{,3}}{\neg\Phi}\to I_{,6}\\ \Pi_1\\ \bot\end{matrix}}{\bot}\,CR_{,5}$$

Π_4 is restated as below.

$$\dfrac{\dfrac{\dfrac{\dfrac{[\neg\bot]^5 \quad \dfrac{[\Phi]^7 \quad \begin{array}{c}[\neg T(\ulcorner\Phi\urcorner)]^8\\ \Pi_2\\ \bot\end{array}}{\bot}\,\Phi E_{,8}}{\bot}\,{\rightarrow}E}{\neg\Phi}\,{\rightarrow}I_{,7} \quad \dfrac{\dfrac{\dfrac{\dfrac{[\neg\bot]^5 \quad \dfrac{[\Phi]^6 \quad \begin{array}{c}[\neg T(\ulcorner\Phi\urcorner)]^3\\ \Pi_2\\ \bot\end{array}}{\bot}\,\Phi E_{,3}}{\bot}\,{\rightarrow}E}{\neg\Phi}\,{\rightarrow}I_{,6} \quad \dfrac{[T(\ulcorner\Phi\urcorner)]^1}{\Phi}\,TE}{\bot}\,{\rightarrow}E}{\Phi}\,\Phi I_{,1}}{\bot}\,{\rightarrow}E}{\bot}\,CR_{,5}$$

Since Π_4 has a maximum formula $\neg\Phi$, we apply $\rhd_{\rightarrow}$ to Π_4 twice and then the derivation has a maximum formula Φ. Then, by applying $\rhd_{\Phi}$ twice, we have the derivation Π_5 below.

$$\dfrac{\dfrac{[\neg\bot]^5 \quad \dfrac{\dfrac{\dfrac{[\neg\bot]^5 \quad \dfrac{\dfrac{[T(\ulcorner\Phi\urcorner)]^1}{\Phi}\,TE \quad \begin{array}{c}[\neg T(\ulcorner\Phi\urcorner)]^3\\ \Pi_2\\ \bot\end{array}}{\bot}\,\Phi E_{,3}}{\bot}\,{\rightarrow}E}{\neg T(\ulcorner\Phi\urcorner)}\,{\rightarrow}I_{,1} \quad \dfrac{\dfrac{\begin{array}{c}[\neg\Phi]^2\\ \Pi_1\\ \bot\end{array}}{\Phi}\,CR_{,2}}{T(\ulcorner\Phi\urcorner)}\,TI}{\bot}\,{\rightarrow}E}{\bot}\,{\rightarrow}E}{\bot}\,CR_{,5}$$

To eliminate the maximum formulas $\neg T(\ulcorner\Phi\urcorner)$ and $T(\ulcorner\Phi\urcorner)$, we apply $\rhd_{\rightarrow}$ and $\rhd_{T(x)}$ in order. Then, we have the derivation Π_6 below.

$$\dfrac{\dfrac{[\neg\bot]^5 \quad \dfrac{[\neg\bot]^5 \quad \dfrac{\dfrac{\begin{array}{c}[\neg\Phi]^2\\ \Pi_1\\ \bot\end{array}}{\Phi}\,CR_{,2} \quad \begin{array}{c}[\neg T(\ulcorner\Phi\urcorner)]^3\\ \Pi_2\\ \bot\end{array}}{\bot}\,\Phi E_{,3}}{\bot}\,{\rightarrow}E}{\bot}\,{\rightarrow}E}{\bot}\,CR_{,5}$$

Π_6 includes the same derivaiton with Π_3. Again Π_3 can be further reduced. Then, we have the following infinite reduction sequence.

$$
\dfrac{
\dfrac{
[\neg\bot]^5 \qquad
\dfrac{
[\neg\bot]^5 \qquad
\dfrac{
\dfrac{
[\neg\bot]^9 \qquad
\dfrac{
[\neg\bot]^9 \qquad
\begin{matrix}
\dfrac{
\dfrac{
[\neg\bot]^i \qquad
\dfrac{
[\neg\bot]^i \quad \overset{\vdots}{\bot}
}{\bot}\to E
}{\bot}\to E
}{\bot}CR_{,i} \\
\vdots \\
\bot
\end{matrix}
}{\bot}\to E
}{\bot}\to E
}{\bot}CR_{,9}
}{\bot}\to E
}{\bot}\to E
}{\bot}CR_{,5}
$$

where $i = 4j+1 (j > 0)$. Therefore, Π_3 initiates a non-terminating reduction sequence and is irreducible with respect to $\mathbb{R}_{CL}$. □

The derivation Π_3 initiates a non-terminating reduction sequence which is not so much a looping reduction as a spiraling reduction. The problem is that Π_3 formalizes the Liar paradox. Since the Liar paradox is a self-referential paradox, Proposition 3.1 shows that *TCSP* is false.

- An additional reduction procedure for *classical reductio* introduced by Gunnar Stålmark (1991, pp. 131-132).

$$
\dfrac{\dfrac{\begin{matrix}[\neg\varphi]^1 \\ \mathfrak{D}_1 \\ \bot\end{matrix}}{\varphi}\,CR_{,1} \qquad \begin{matrix}\mathfrak{D}_2 \\ \psi\end{matrix}}{\sigma}\circ E
\qquad \rhd_{CR(\circ)} \qquad
\dfrac{\begin{matrix}\dfrac{[\neg\sigma]^2 \qquad \dfrac{[\varphi]^1 \qquad \begin{matrix}\mathfrak{D}_2 \\ \psi\end{matrix}}{\sigma}\circ E}{\bot}\rightarrow I_{,1} \\ \mathfrak{D}_1 \\ \bot\end{matrix}}{\sigma}\,CR_{,2}
$$

HAPPY WORLD LOGIC DAY!!