Strong Types and the Lascar group

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References

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Preliminaries : Monster Model

Fix a first order language $\mathcal{L}$, complete theory T.

Definition

A set of $\mathcal{L}$-formulas is called a **(complete) type** if it is a maximal consistent set of $\mathcal{L}$-formulas.

We fix a huge model $\mathcal{M}$ of T which satisfies :

1. Any 'small' set or tuple of elements we mention are in $\mathcal{M}$.
2. Any (consistent) type is realized in $\mathcal{M}$.
3. Any partial isomorphism into itself $\mathcal{M}$ is extended to an automorphism.
4. Any small model M of T can be regarded as an elementary substructure of $\mathcal{M}$ ($M \prec \mathcal{M}$).

Strong Types

Let a, b be any small tuples of elements.

Definition

1. $a \equiv b$: a and b have the same **type** iff for any $\mathcal{L}$-formula $\varphi(x)$, $a \models \varphi(x)$ iff $b \models \varphi(x)$.
2. $a \equiv^{\mathrm{s}} b$: a and b have the same (Shelah)-**strong type** iff for any definable equivalence relation E having finitely many classes, $E(a, b)$ holds.
3. $a \equiv^{\mathrm{KP}} b$: a and b have the same **KP-type** iff for any bounded type-definable equivalence relation E, $E(a, b)$ holds.
4. $a \equiv^{\mathrm{L}} b$: a and b have the same **Lascar type** iff for any bounded (automorphism-)invariant equivalence relation E, $E(a, b)$ holds.

$\equiv \ < \ \equiv^{\mathrm{s}} \ < \ \equiv^{\mathrm{KP}} \ < \ \equiv^{\mathrm{L}}$, where $<$ means 'is coarser than'.

Interpretation in terms of automorphisms

Proposition

The following are equivalent :

1. $a \equiv b$: *a and b have the same* ***type***.
2. *For any* $\mathcal{L}$*-formula* $\varphi(x)$, $a \models \varphi(x)$ *iff* $b \models \varphi(x)$.
3. *There is* $f \in \mathrm{Aut}(\mathcal{M})$ *such that* $f(a) = b$.

Interpretation in terms of automorphisms

Proposition

The following are equivalent :

1. $a \equiv^{s} b$: *a and b have the same (Shelah)-**strong type**.*
2. *For any definable equivalence relation E having finitely many classes, $E(a, b)$ holds.*
3. *There is $f \in \mathrm{Aut}(\mathcal{M})$ such that f pointwise fixes $\mathrm{acl}^{\mathrm{eq}}(\emptyset)$ and $f(a) = b$.*

($\mathrm{acl}^{\mathrm{eq}}(\emptyset)$ = the set of definable equivalence classes whose number of automorphic images is finite.)

Interpretation in terms of automorphisms

Proposition

The following are equivalent :

1. $a \equiv^{\mathrm{KP}} b$: *a and b have the same **KP-type**.*
2. *For any bounded type-definable equivalence relation E, $E(a,b)$ holds.*
3. *There is $f \in \mathrm{Aut}(\mathcal{M})$ such that f pointwise fixes $\mathrm{bdd}(\emptyset)$ and $f(a) = b$.*

($\mathrm{bdd}(\emptyset) =$ the set of type-definable equivalence classes whose number of automorphic images is bounded.)

Indiscernible sequences and Lascar strong automorphisms

For the Lascar type, there are more interesting characterizations. Before stating them, we need some definitions.

Definition

Let I be any linearly ordered set. A sequence $(a_i : i \in I)$ is called an **indiscernible sequence** if $a_{i_0} \cdots a_{i_n} \equiv a_{j_0} \cdots a_{j_n}$ for any $i_0 < \cdots < i_n, j_0 < \cdots < j_n \in I$.

Definition

$\mathrm{Autf}(\mathcal{M})$ is the subgroup of $\mathrm{Aut}(\mathcal{M})$ generated by $\{f \in \mathrm{Aut}(\mathcal{M}) : f$ pointwise fixes some small model $M \models T\}$.

Characterization of Lascar types

Theorem

The following are equivalent :

1. $a \equiv^{L} b$: *a and b have the same* ***Lascar type***.
2. *For any bounded invariant equivalence relation* E, $E(a, b)$ *holds.*
3. *There is* $f \in \mathrm{Autf}(\mathcal{M})$ *such that* $f(a) = b$.
4. *There are* $c_0, \cdots c_n$ *with* $c_0 = a, c_n = b$ *such that for each* $0 \leq k \leq n - 1$, *there is an indiscernible sequence* I_k *such that* $c_k, c_{k+1} \in I_k$.

The Lascar group : Group

Definition

$\mathrm{Gal_L}(T) = \mathrm{Aut}(\mathcal{M})/\,\mathrm{Autf}(\mathcal{M})$ is the **Lascar group** of T.

Remark

The Lascar group does not depend on the choice of a monster model up to isomorphism.

The Lascar group : Topology

Definition

Let A be any small subset of $\mathcal{M}$. $S(A) = \{\text{complete types over } A\}$.

Equip topology given by basic open sets of the form
$[\varphi(x)] = \{p \in S(A) : \varphi(x) \in p\}$.

Proposition

$S(A)$ is a Stone space. i.e. compact totally separated space.

The Lascar group : Topology

Let M be any small model.

Definition

$S_M(M) = \{\mathrm{tp}(f(M)/M) : f \in \mathrm{Aut}(\mathcal{M})\} \subseteq S(M)$.

Equip $S_M(M)$ with subspace topology.
Define $\nu : S_M(M) \to \mathrm{Gal_L}(T)$ by $\nu(\mathrm{tp}(f(M)/M)) = f \cdot \mathrm{Autf}(\mathcal{M})$.
ν is well-defined and we give quotient topology on $\mathrm{Gal_L}(T)$.

Proposition

The topology of $\mathrm{Gal_L}(T)$ *does not depend on the choice of a small model* M*.*

Theorem

$\mathrm{Gal_L}(T)$ *is a quasi-compact topological group.*

Two subgroups of $\mathrm{Gal}_{\mathrm{L}}(T)$

Let $\mathrm{Gal}^0_{\mathrm{L}}(T)$ be the connected component of $\mathrm{Gal}_{\mathrm{L}}(T)$ containing $\{\mathrm{id}\}$.

Remark

Being a topological group, $\overline{\{\mathrm{id}\}}$ and $\mathrm{Gal}^0_{\mathrm{L}}(T)$ are both closed normal subgroups of $\mathrm{Gal}_{\mathrm{L}}(T)$.

Interpretation in terms of orbit equivalence relation

Definition

Let $\pi : \mathrm{Aut}(\mathcal{M}) \to \mathrm{Gal_L}(\mathcal{M})$ be the projection map and H a subgroup of $\mathrm{Gal_L}(T)$. Define an **orbit equivalence relation** $\equiv^H$ by $a \equiv^H b$ iff there is $f \in \pi^{-1}(H)$ such that $f(a) = b$.

Theorem

$\equiv^H$ is type-definable iff H is closed in $\mathrm{Gal_L}(T)$.

Theorem

For any small tuples $a, b \in \mathcal{M}$,

1. $a \equiv b$ *iff* $a \equiv^{\mathrm{Gal_L}(T)} b$,
2. $a \equiv^{\mathrm{s}} b$ *iff* $a \equiv^{\mathrm{Gal_L^0}(T)} b$,
3. $a \equiv^{\mathrm{KP}} b$ *iff* $a \equiv^{\overline{\{id\}}} b$,
4. $a \equiv^{\mathrm{L}} b$ *iff* $a \equiv^{\{id\}} b$.