The Meanings of Logical Constants and Identity

A Proof-theoretic and Verificationist Account

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Korea Logic Day, 2021.01.14

Contents

0. What is a logical constant?

- Some Criteria: topic-neutrality, universal applicability.
- How is the meaning of a logical constant to be explained?
- Truth-conditional vs. Verificationist account of the meanings of the intuitionist logical constants.

1. A verificationist, proof-theoretic account of the meanings of the logical constants of FOL (in line with Gentzen, BHK, Dummett, Prawitz, Martin-Löf).

2. How is the meaning of identity to be explained?

- Can it be defined in terms of the second-order quantifiers?
- A verificationist, proof-theoretic account of the meaning of identity: The canonical proofs for identity propositions, I and E rules for identity.
- Circularity in the usual explanation of identity and the distinction between definitional equality and identity proposition.
- Type-dependency of identity and of object.

What is Logic?

- Logic as a science began with the works by Aristotle (384-322 B.C.): *Organon* (*Categories, On Interpretation, Prior Analytics, Posterior Analytics, Topics, On Sophistical Refutations*).
- Kant thought Aristotle's logic is essentially complete. But a large part of mathematical reasoning cannot be adequately deal with in Aristotle's syllogism.
- With the philosophical aim for reducing arithmetic to logic, G. Frege (1848-1925) tried to show that arithmetical reasoning does not depend on extra-logical intuition, pure or empirical, giving birth to a renewed beginning for the science of logic: *Begriffschrift* (1879), *Grudgesetze der Arithmetik Vol. I*(1893), *Vol II* (1903).
- Logic is primarily concerned with inferences and arguments.
- The correctness or validity of an inference or an argument is a matter of form. (So logic is 'formal'.)
- Logical forms of arguments are determined by the logical constants.

What is a logical constant?

- Logical notions are topic-neutral and laws of logic are universally applicable. (G. Frege)
- A way of capturing topic-neutrality: Permutation invariance.

  (A. Tarski (1986), "What are Logical Notions?")
- Logical notions are insensitive to the particular identities of objects, or more precisely, invariant under arbitrary permutations of domains of objects.
- Under a development of this criterion, the usual sentential connectives, (the first-order and other) quantifiers, the identity relation and the notion of an object (thing) are logical constants, while the basic mathematical notions such as a natural number and the membership relation are not.

What is a logical constant?

- The universal applicability of logic: logic is a universal canon for reasoning, one that is applicable not just to reasoning about this or that domain, but to all reasoning.
- As logic is primarily concerned with inferences rather than truths, logical constants and logical laws may be more appropriately captured by inference rules rather than axioms.

(So Gentzen's natural deduction system may be more appropriate for analyzing logical inferences than Frege-Hilbert's axiomatic system)

- Logical constants may be characterized by 'purely inferential' introduction and elimination rules.

What are the meanings of logical constants?

- The validity of an argument depends on the meanings of logical constants.
- How are the meanings of logical constants to be explained?

(Truth-conditional, verificationist, pragmatist and other inferentialist theories of meaning)

- Classical logic and the Truth-conditional theory of meaning

- The meaning of a sentence is determined by its (bivalent) truth-condition.
- The meaning of a logical constant is explained in terms of its role for determining the truth condition of sentences containing it.
- The classical bivalent conception of truth is verification-transcendent.
- It is doubtful how a verification-transcendent truth-condition is knowable and manifestable, if knowable. (M. Dummett)

● Intuitionist logic and the Verificationist theory of meaning

- The meaning of a sentence is determined by its (canonical) verification condition.
- The meaning of a logical constant is explained in terms of its role for determining the (canonical) verification-condition of sentences containing it.
- Verifications or proofs of a sentence amount to methods for obtaining a canonical verification or proof of the sentence.
- BHK explanation for the meanings of logical constants and Gentzen's introduction rules can be understood as specifying canonical proofs of compound propositions of the (intuitionistic) first-order logic.
- Rules of Martin-Löf Type theory (ITT) explicitly specify canonical proofs of logically compound propositions.

● BHK explanations (A. S. Troelstra & D. van Dalen (1988))

- A proof of $A \wedge B$ is given by presenting a proof of A and a proof of B.
- A proof of $A \vee B$ is given by presenting either a proof of A or a proof of B (plus the stipulation that we want to regard the proof presented as evidence for $A \vee B$).
- A proof of $A \rightarrow B$ is a construction which permits us to transform any proof of A into a proof of B.
- Absurdity $\perp$ (contradiction) has no proof; a proof of $\neg A$ ($A \rightarrow \perp$) is a construction which transforms any hypothetical proof of A into a proof of a contradiction.
- A proof of $\forall x A(x)$ is a construction which transforms a proof of $d \in D$ (D the intended range of the variable x) into a proof of A(d).
- A proof of $\exists x A(x)$ is given by providing $d \in D$, and a proof of A(d).

Gentzen & some inferentialist's explanations

- Gentzen's explanation of the meanings of (intuitionistic) logical constants: Introduction rules "represent, as it were, the 'definitions' of the symbols concerned, and the eliminations are no more, in the final analysis, than the consequences of these definitions" (1935, §5.13; 1969, 80).
- In so far as the rules of the intuitionist logic (NJ) are concerned, Gentzen's explanation essentially agrees with BHK and the verificationist accounts of logical constants.
- A. N. Prior's 'tonk' against "analytic validity" or an inferentialist theory.

$$\frac{A}{A \text{ tonk } B} \qquad \frac{A \text{ tonk } B}{B}$$

- (Tonk E) cannot be justified by, or a 'consequence of' in Gentzen's words, the 'meaning' of tonk given by (tonk I). (Tonk E) is not in harmony with (Tonk I) in Dummett's words; they do not satisfy Prawitz's inversion principle.

Harmony and Inversion Principle

- Dummett's Principle of Harmony: Our linguistic practice, to be coherent, requires a harmony between the grounds and the consequences of an assertion.
- E.g. Whatever is taken to be a consequence of an assertion must be justifiable by the conclusive grounds for the assertion.
- Prawitz's inversion principle (1965) generalized: A canonical deduction, i.e. a deduction ending with an application of an I-rule of the major premise of an E-rule, with the deductions for its minor premises already contain a deduction for the conclusion of the E-rule.
- I and E rules satisfying the inversion principle generates a reduction procedure, which may be viewed as a justification of the E rules in terms of the corresponding I rules.

Reduction procedures and Normalization

E.g.

Σ_1 A $\quad\quad$ (A) $\quad$ (B) ——— $\quad$ Σ_2 $\quad$ Σ_3 $A \vee B$ $\quad\quad$ C $\quad$ C ———————— C	Σ_1 A Σ_2 C	Σ_1 $A[x/t]$ $\quad\quad$ $(A[x/a])$ ——— $\quad\quad$ Σ_2 $\exists x A$ $\quad\quad$ C ———————— C	Σ_1 $A[x/t]$ $\Sigma_2[a/t]$ C
$\vee$ reduction		$\exists$ reduction	

Normalization Theorem: Every deduction in NJ can be reduced to a normal form, i.e. a deduction without maximum formulas. (Prawitz (1965))

Cf. Gentzen's Cut Elimination theorem.

Missing rules for identity

- Gentzen’s NJ does not contain rules for identity.
- Is identity a logical notion?
- How is the meaning of identity to be explained?
- Frege in *Begriffschrift* (1879) held the identity relation to be a relation between linguistic expressions (names), but he correctly abandoned this view in “Über Sinn und Bedeutung”(1892).
- The notion of identity has important consequences for semantics, metaphysics and epistemology.
- We shall attempt a verificationist proof-theoretic account of identity and exhibit it’s relation with the notion of object.

Second-order quantifiers

- Why not define t=u as $\forall X(Xt \leftrightarrow Xu)$?

  Cf. Leibniz's principle of identity of indiscernibles.
- The definition is supposed to be given in the standard, impredicative second-order logic, where the range of second-order quantifier includes impredicatively defined subsets, i.e. subsets defined by an indispensable use of a second-order quantifier, of a domain of the first-order quantifier.
- From the verificationist viewpoint, it can be argued that the uses of impredicative second-order quantifiers are illegitimate and hence, the alleged definition of identity based on the impredicative universal second-order quantifier is also improper.

Rules for second-order quantifiers

($\forall^2$I)

$$\frac{A[X^n/Y^n]}{\forall X^n A}$$

($\exists^2$I)

$$\frac{A[X^n/\lambda x_1,\ldots,x_n B]}{\exists X^n A}$$

($\forall^2$E)

$$\frac{\forall X^n A}{A[X^n/\lambda x_1,\ldots,x_n B]}$$

($\exists^2$E)

$$\frac{\exists X^n A \qquad \begin{matrix}(A[X^n/Y^n])\\ C\end{matrix}}{C}$$

with appropriate restrictions on Y^n in ($\forall^2$I) and ($\exists^2$E).

Are the uses of second-order quantifiers harmonious?

$\sum$ $A[X^n/Y^n]$ _____________ $\forall^2 I$ $\forall X^n A$ ______________ $\forall^2 E$ $A[X^n/\lambda x_1,\ldots x_n B]$	$\sum[Y^n/\lambda x_1,\ldots x_n B]$ $A[X^n/\lambda x_1,\ldots x_n B]$	$\sum_1$ $A[X^n/\lambda x_1,\ldots x_n B]$ $(A[X^n/Y^n])$ _____________ $\exists^2 I$ $\sum_2$ $\exists X^n A$ C __________________________ $\exists^2 E$ C	$\sum_1$ $A[X^n/\lambda x_1,\ldots x_n B]$ $\sum_2[Y^n/\lambda x_1,\ldots x_n B]$ C
$\forall^2$ reduction		$\exists^2$ reduction	

No!

- From the verificationist viewpoint, in order for the ($\forall^2$ I) to have the role of fixing the meaning of $\forall^2$ by specifying the canonical verification of $\forall X^n A$, the range of values of the free variable Y^n in the premise cannot contain properties or sets definable only in terms of $\forall^2$.
- Otherwise, ($\forall^2$ I) would be circular as a specification of the meaning of $\forall^2$ and cannot play the role of meaning-determining introduction rule.
- On the other hand, the standard impredicative use of ($\forall^2$ E) is such that B in the conclusion may essentially contain $\forall^2$.
- So, in so far as ($\forall^2$ I) is understood as an introduction rule giving the meaning of $\forall^2$, the standard ($\forall^2$ E) is not harmonious with it.
- Hence, the alleged definition of identity in terms of the impredicative quantifier is not acceptable from the verificationist viewpoint.
- Note: the reduction procedure for the second-order quantifiers does not guarantee a reduction of the degree of maximum formula, and normalization does not hold for the standard second-order logic.

Frege on identity

- Frege did not define identity in terms of second-order quantifiers. Frege held the identity relation to be primitive, being governed by two axioms: Reflexivity (Every object is identical to itself) and the indiscernibility of identicals (If an object is identical to another, then whatever holds of the former holds of the latter.)
- Gentzen style ND rules corresponding to Frege's axioms for identity

(Reflexivity)

$$\frac{}{t=t}$$

(Indiscernibility of Identicals)

$$\frac{t=u \qquad \varphi(t)}{\varphi(u)}$$

(=I) and (=E) rules?

- Do (Reflexivity) and (Indiscernibility of Identicals) play the role of identity introduction and elimination rules?
- Some authors think not: “But we cannot replace Frege’s axiom of identity by paired rules of inference (or development) which provide between them for the introduction and the elimination of the identity signs.” (W. Kneale & M. Kneale (1962) p.742)
- We shall argue that (Reflexivity) can be regarded as the meaning-giving identity introduction rule in the context of FOL and this determines the identity elimination rule, which is equivalent with (Indiscernibility of Identicals) in FOL.

A natural explanation of identity

- Everything is identical to itself and only to itself.
- The identity relation is the relation which every object holds to itself and only to itself.
- A binary relation R is an identity relation iff every object (i) holds of R to itself and (ii) only to itself.
- The identity relation is the least reflexive relation.
- A natural formulation of these two conditions as a definition of identity relation is circular.
- This is because it seems that the identity relation is presupposed explicitly in the condition (ii), $\forall x \forall y(Rxy \rightarrow x=y)$, and implicitly in the condition (i).
- The implicit presupposition in (i) is in the presupposition of the notion of object: it needs a criterion of identity. (Cf. Frege)

(=I) and (=E)

- The explicit circularity may be avoided by a proof-theoretic approach.
- The implicit circularity may be avoided by distinguishing definitional equality and identity proposition (type).
- Following Martin-Löf (1972), identity as the least reflexive relation may be characterized proof-theoretically by the following pair of rules in the language of the first-order logic.

(=Introduction)

$$\frac{}{t=t}$$

(= Elimination)

$$\frac{t=u \quad \psi(x,x)}{\psi(t,u)}$$

Forms of identity introduction rule

- In order to conclude that t=t, the only premise needed is that t is a thing, an object. So a general form of = Introduction rule may be formulated as:

$$\frac{\text{t is an object}}{t=t}$$

- For Frege, an object is the reference of a proper name (singular term) and both notions - object and proper name - are primitive. In FOL, it is presupposed that every name has a unique reference. So (=I) in FOL may be formulated simply:

$$\frac{\quad}{t=t}$$

- If every object is of a certain type, and every identity proposition is of the form 't is the same A as u' ($t=_A u$) then (=I) in ITT obtains:

$$\frac{A\ \text{type} \qquad t: A}{\text{refl}(t): t=_A t}$$

Interderivability of (=E) and (Indiscerniblity of Identicals)

- (Indiscernibility of Identicals) is derivable from (=E) by letting $\psi(x,x)$ to be $\Phi(x)\rightarrow\Phi(x)$, using (→I) and (→E). (The converse also holds by letting $\Phi(t)$ to be $\Psi(t,t)$.)

$$\dfrac{\Phi(t) \qquad \dfrac{t=u \qquad \dfrac{[\Phi(x)]}{\Phi(x)\rightarrow\Phi(x)}\,(\rightarrow I)}{\Phi(t)\rightarrow\Phi(u)}\,(=E)}{\Phi(u)}\,(\rightarrow E)$$

(=E) and (Indiscernibility of Identicals) are in harmony with (=I):
They satisfy the inversion principle.

$$\dfrac{\dfrac{}{t=t}(=I) \qquad \begin{matrix}\Sigma \\ \psi(x,x)\end{matrix}}{\psi(t,t)}(=E) \qquad \underset{=\text{reduction}}{\rightsquigarrow} \qquad \begin{matrix}\Sigma[x/t] \\ \psi(t,t)\end{matrix}$$

$$\dfrac{\dfrac{}{t=t}(=I) \qquad \begin{matrix}\Sigma \\ \varphi(t)\end{matrix}}{\varphi(t)}(II) \qquad \underset{\text{simplification}}{\rightsquigarrow} \qquad \begin{matrix}\Sigma \\ \varphi(t)\end{matrix}$$

Is (=I) general enough?

- The major premise of (=E) and of (Indiscenibility of Identicals) is of the form t=u, while the conclusion of (=I) is of the form t=t. So, it may seem that the significance of =reduction is limited and it might be suggested that a more general = Introduction rule with a conclusion of the form t=u, or the canonical proofs of propositions of the form t=u should be given.
- A typical way of introducing propositions of the form t=u depends on definitions.
- Distinguish definitional equivalence (≡) and identity (=)

(Definitional equivalence should be effectively decidable, but identity need not.)

Definitional equivalence

≡ is an equivalence relation and the following rule holds:

(Def Sub)

$$\frac{\varphi(\alpha) \qquad \alpha \equiv \beta}{\varphi(\beta)}$$

(Def Sub) should be distinguished from (Indiscernibility of Identicals): t=u is derivable from t≡u by (=I) and (Def Sub) but the converse does not hold.

(Def Sub) as a means of canonical proof?

- It has been suggested that (Def Sub) can be used in any canonical proof. (A. Klev (2019).) In particular, it has been suggested that derivations of the following form are canonical:

$$\frac{\qquad\qquad}{t=t}\ (=\mathrm{I})$$

$$\frac{t=t}{t=u}\ \text{(Def Sub)}$$

Such an extension of canonical proofs for identity propositions is not needed.

<table>
<tr><td colspan="2">(1)</td><td colspan="2">(2)</td><td>(3)</td></tr>
<tr>
<td colspan="2">________(=I)
t(a)=t(a)   a≡b
________________(DS)   ∑
t(a)=t(b)   ψ(x,x)
________________________________(=E)
ψ(t(a),t(b))</td>
<td colspan="2">________(=I)   ∑
t(a)=t(a)   ψ(x,x)
______________________(=E)
ψ(t(a),t(a))   a≡b
______________________________(DS)
ψ(t(a),t(b))</td>
<td>∑[x/t(a)]
ψ(t(a),t(a))   a≡b
___________________(DS)
ψ(t(a),t(b))</td>
</tr>
<tr><td colspan="3">permutation: (1)⇝(2)</td><td colspan="2">= reduction: (2)⇝(3)</td></tr>
</table>

E.g. When (II) is regarded as an = Elim rule:

<table>
<tr><td>(1)</td><td>(2)</td><td>(3)</td></tr>
<tr>
<td>____(=I)
5=5   3+2≡5
______________(DS)   ∑
3+2=5   5 is prime
____________________________(II)
3+2 is prime</td>
<td>____(=I)   ∑
5=5   5 is prime
____________________(II)
5 is prime   3+2≡5
________________________________(DS)
3+2 is prime</td>
<td>   ∑
5 is prime   3+2≡5
______________________(DS)
3+2 is prime</td>
</tr>
<tr><td colspan="2">permutation: (1)⇝(2)</td><td>simplification: (2)⇝(3)</td></tr>
</table>

Type dependency of canonical proofs for identity propositions

- The extension of canonical proofs for identity propositions by (Def Sub) is not only unnecessary but also misguided.
- There can be ways of proving or verifying propositions of the form t=u other than via (Def Sub).
- Identity propositions of the form t=t may be asserted in so far as one knows t is an object of some kind or type A without knowing what kind or type A is.
- In order to assert identity propositions of the form t=u, one must know the condition for objects t and u to be identical.
- The identity condition for t and u depends on what kind or type of objects t and u are. E.g. numbers, figures, particles and persons have different identity conditions.
- So in order to assert identity propositions of the form t=u, more properly $t =_A u$, one must know what kind or type A is.
- It follows that there is no type-independent canonical proof for t=u.

Type-dependency of identity

- As the proof conditions for identity propositions of the form "t and u are the same"(t=u, I(t,u)) differ depending on what kind (type) of objects t and u are, the proper form of identity propositions should be "t and u are the same A"($t=_A u$, I(A,t,u)), where A is the type of t and u, according to the intuitionist or verificationist account.
- According to such an account, in usual statements of the form t=u, an indication of the type of t and u is implicit in context and can be explicitly recovered.
- Sortals and types carry application condition and identity condition.
- In order to specify a type, one must specify what counts as canonical objects of the type and when objects of the type are identical.
- As the identity condition of objects of a type constitutes part of the definition of the type, the identity relation used for the identity condition of a type is definitional, which should be distinguished from identity propositions.

Definitional identity and identity proposition in ITT

- The distinction is made explicit in ITT as identity judgements of the form t=u : A and identity proposition $t=_A u$ (or I(A,t,u)).

(=I)	(=E)
A type, t: A ——————— refl(t): $t=_A t$	t:A, u:A, a:$t=_A u$ C(x,y,z) type [x:A, y:A, z:$x=_A y$] c(x): C(x,x,refl(x)) [x:A] ———————————————— ide(a,c): C(t,u,a)

Definitional identity and identity proposition in ITT

- Objects of a type are identical if they reduce to a same canonical object of the type; ITT has a second introduction rule for each type, specifying the identity condition for the canonical objects of the type introduced (the first introduction rule, which specifies the canonical objects of the type introduced, corresponds to the application condition of a sortal.)
- E.g.

(NI)	(NI)′
$\dfrac{}{0:N}$ $\dfrac{t:N}{s(t):N}$	$\dfrac{}{0=0:N}$ $\dfrac{t=u:N}{s(t)=s(u):N}$

Object and Identity

- (=I) specifies the canonical proofs of identity propositions.
- According to (extended) Gentzen's and the verificationists' view, (=I) may be regarded as a (partial) explanation of the meaning of identity in terms of its canonical verification.
- The explanation says the canonical proof of an identity proposition is obtained whenever an object (of a type) is given. This is implicit in "every object (of any type) is identical to itself"
- It was suspected that the explanation is circular because the notion of object presupposes a notion of identity as the notion of object requires identity condition.
- The notions of object and identity are inter-related. Corresponding to the thesis of the type-dependency of identity is type-dependency of object.
- Type-dependency of object: every object is an object of certain type.

(An Object is always an object of a certain type. There is no 'bare' object.)

- The alleged circularity may be avoided by distinguishing definitional identity and propositional identity as in ITT: the notion of "objects of type A" depends on the identity condition for objects of type A, which is definitional but not propositional.

Some further implications of the theses regarding identity and object.

- If an object is always an object of a certain type then it may be argued that a quantification over absolutely all objects could become dubious and the range of any quantifier should be restricted to a type of objects.
- The thesis of type-dependency of identity may be compared with Peter Geach's relative identity thesis. But although the two theses look superficially similar, some of Geach's central claims, for example, "there are kinds F and G such that some of the same F's are not the same G's", seem to be in tension with the type-dependency of identity thesis.
- The thesis of type-dependency of identity does not imply that the usual notion of identity is ambiguous or that there are as many notions of identity as there are types. On the contrary, from the verificationist viewpoint, (=I) gives the meaning of identity by a uniform specification of the canonical proof of an identity proposition of any type. The (=I) and (=E) rules in effect say that for any type A, $=_A$ is the least reflexive relation over objects of type A .

Thank You!

References

정인교 (2005) 「조화와 비서술성」, 『철학적분석』 제 12호, 99-120.

정인교 (2019) 「동일성의 의미」, 『철학』141집, 249-278.

Dummett, M. (1981) *Frege: Philosophy of Language* 2nd ed. Duckworth.

Dummett, M. (1991) *The Logical Basis of Metaphysics,* Harvard Univ. Pr.

Frege, G. (1879) *Begriffschrift*, Halle.

Frege, G. (1892) "Über Sinn und Bedeutung" *Zeitscrift für Philosophie und philosophische Kritik*, 100: 25-50. Trans. as "On Sense and Reference" in Geach, P. and Black, M. *Translations from the Writings of Gotlob Frege*, pp. 56-78, Blackwell.

Geach, P. (1980) *Reference and Generality*, 3rd edn. Cornell Univ. Pr.

Gentzen, G. (1969) *The Collected Papers of Gebhard Gentzen*, (ed. by M. Szabo), North-Holland.

Kneale, W. and M. Kneale, (1962) The Development of Logic

Klev, A. (2019) "The Harmony of Identity", *Journal of Philosophical Logic.* Published online 12 January. (https://doi.org/10.1007/s10992-018-09499-0)

Martin-Löf, P. (1971) "Hauptsatz for the intuitionistic theory of iterated inductive definitions". In Fenstad, J.E. (ed.) *Proceedings of the Second Scandinavian Logic Symposium,* pp. 179-216, North-Holland.

Nordström, B. Petersson, K. and Smith, J. (1990) *Programming in Martin-Löf's Type Theory*, Clarendon Pr.

Prawitz, D. (1965) *Natural Deduction,* Almqvist & Wiksell.

A. Tarski (1986), “What are Logical Notions?”, *History and Philosophy of Logic* 7, pp. 143-54.

Troelstra, A. S. and D. van Dalen, (1988) *Constructivism in Mathematics: An Introduction*, Vol I.